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A multiplier approach for approximating and estimating extreme of compound distributions

Helgard Raubenheimer

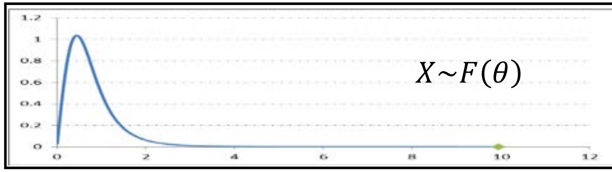
Riaan de Jongh, Tertius de Wet, Fred Lombard

Centre for Business Mathematics and Informatics ®

Layout of presentation

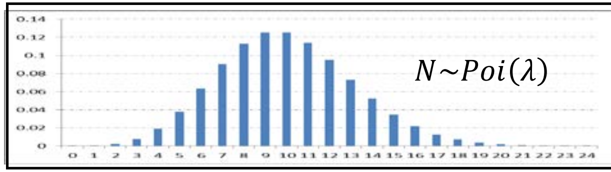
- Compound distributions and quantile estimation
- Theoretical framework and approximation methods
- Simulation study results

Theoretical framework



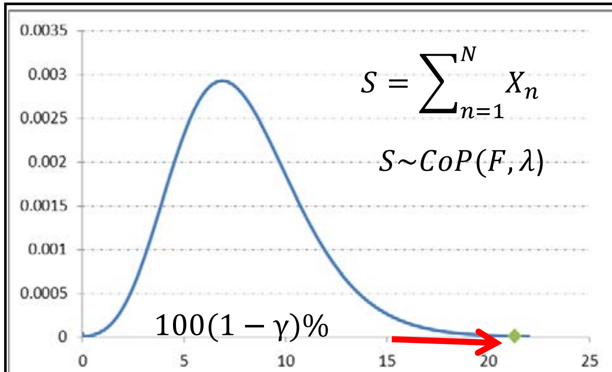
Let $X_1, \ldots, X_N \sim F$

Loss severity distribution



$N \sim Poi(\lambda)$

Frequency distribution



$S = \sum_{n=1}^{N} X_n \sim CoP(F, \lambda) = G$

Aggregate loss distribution

We are interested in estimating the $100(1-\gamma)\%$ *quantile of compound distribution, typically used for capital.*

Approximation methods

Approximating the distribution

1) Monte Carlo (MC) approach

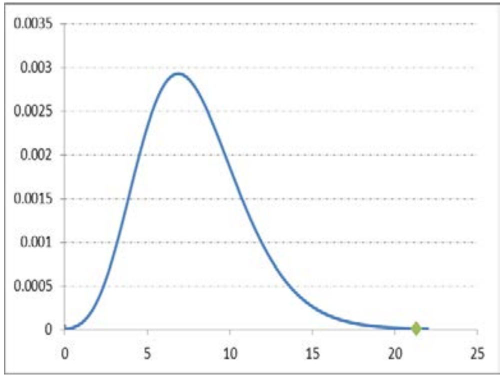
2) Numerical approximation alternatives:

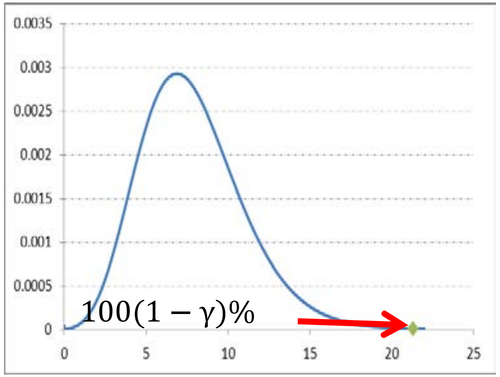
- Panjer recursion (Panjer, 1981)
- Fast Fourier transforms (Meyers,1983)

Approximating the quantile (closed-form)

1) Single loss approximations

2) Perturbative approaches





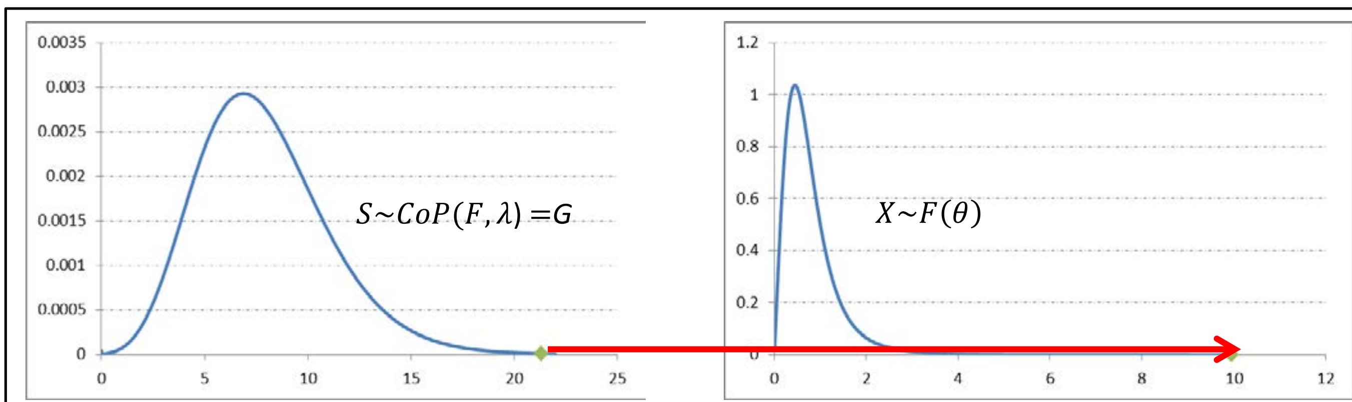
Approximating Quantile: Closed form

Single Loss Approximations (SLA) - Böcker and Klüppelberg (2005) :

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right)$$

Perturbative approximation (PA) - Hernandez et al. (2014) :

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1+\frac{\ln(1-\gamma)}{\lambda}\right)$$



Approximating Quantile: Closed form

SLAD: Degen (2010), using second order exponentiality, derived an improved single loss approximation

- *finite mean* ($\mu = E(X) < \infty$)

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right) + \lambda\mu,$$

- *infinite mean models* ($\mu = E(X) = \infty$)

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right) + \gamma F^{-1}\left(1-\frac{\gamma}{\lambda}\right)\frac{C_\kappa}{1-\frac{1}{\kappa}}$$

where $C_\kappa = (1-\kappa)\frac{\Gamma^2\left(1-\frac{1}{\kappa}\right)}{2\Gamma\left(1-\frac{2}{\kappa}\right)}$, Γ the gamma function, $\kappa = \mathrm{EVI}$

Approximating Quantile: Closed form

Hernandez et al. (2014) introduced k-th order perturbative approximations for calculating $G^{-1}(1-\gamma)$ of the compound loss distribution. $Q^{(K)} = Q_0 + \sum_{k=1}^{K} \frac{1}{k!} Q_k$

The perturbative correction Q_k are small if $X_{[N]} \gg \sum_{i=1}^{N-1} X_{[i]}$

$$Q_0 = F^{-1}\left(\frac{\lambda + \ln(1-\gamma)}{\lambda}\right)$$

$$Q_1 = (\lambda + \ln(1-\gamma))E(X|X < Q_0)$$

$$Q_2 = -\left(\lambda f(Q_0) + \frac{f'(Q_0)}{f(Q_0)}\right)(\lambda + \ln(1-\gamma))E(X^2|X < Q_0) - \lambda f(Q_0)\, Q_0^2$$

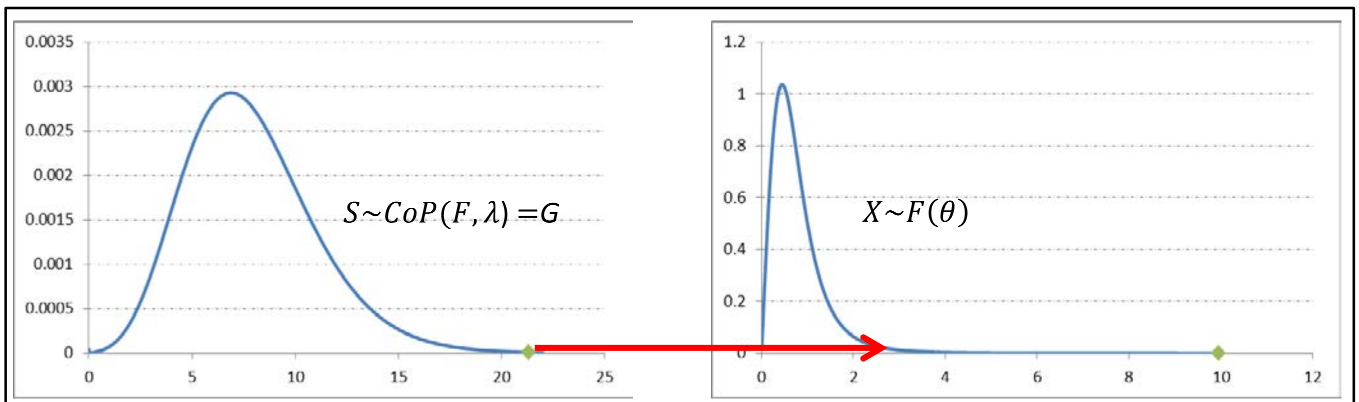
with f the density of F.

Approximating Quantile: Multiplier

Can we approximate the $100(1-\gamma)\%$ quantile of the compound distribution G by using a multiplier (denoted by θ) and the $100(1-\gamma^*)\%$ quantile the distribution of F where $\gamma^* > \gamma$?

i.e. $G^{-1}(1-\gamma) \approx \theta \times F^{-1}(1-\gamma^*)$

$\theta_{SLA} \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}$, $\theta_{PA} \approx \left(\frac{\lambda \ln(1-\gamma^*)}{\ln(1-\gamma)}\right)^{\kappa}$, where $\kappa = EVI$



Extreme value index / sub-exponential

Note that the sub-exponential class includes all regularly varying densities.

A positive measurable function h is regularly varying with parameter β ($h \in RV_\beta$), if

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\beta \text{ for all } x > 0.$$

Extreme value index / sub-exponential

In the case of a probability density f by Karamata's Theorem, if $f \in RV_{-\frac{1}{\kappa}-1}$ then $\bar{F} \in RV_{-1/\kappa}$ where $\bar{F}(x) = 1 - F(x)$ (see e.g. Embrechts et al. 1997).

In terms of its tail quantile function $U(t) = F^{-1}\left(1 - \frac{1}{t}\right)$, this is equivalent to $U \in RV_{\kappa}$.

κ is the extreme value or tail index of the distribution.

Extreme value index / sub-exponential

Regularly varying functions are functions which can be represented by power functions multiplied by slow varying functions, i.e. if $\lim_{t\to\infty} L(tx)/L(t) = 1$ then $h(x) = x^{\beta}L(x)$ where $L(x)$ is a slow varying function.

We write L_F and L_U for the slow varying function associated with $\bar{F}$ and U respectively.

Approximating Quantile: Multiplier

F is regularly varying then, under certain limiting conditions,

$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right)$ (SLA, see e.g. Bocker and Kluppelberg 2005 or Degen 2010)

$$F^{-1}\left(1-\frac{\gamma}{\lambda}\right) = U\left(\frac{\lambda}{\gamma}\right) = \frac{U\left(\frac{\lambda}{\gamma}\right)}{U\left(\frac{1}{\gamma^*}\right)} U\left(\frac{1}{\gamma^*}\right)$$

$$= \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa} \frac{L_U\left(\frac{\lambda}{\gamma}\right)}{L_U\left(\frac{1}{\gamma^*}\right)} U\left(\frac{1}{\gamma^*}\right)$$

$$= \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa} F^{-1}(1-\gamma^*) \frac{L_U\left(\frac{\lambda}{\gamma}\right)}{L_U\left(\frac{1}{\gamma^*}\right)}$$

$$\theta \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}, \ \gamma \to 0$$

i.e. $G^{-1}(1-\gamma) \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa} F^{-1}(1-\gamma^*)$

Approximating Quantile: Multiplier

Similarly for the perturbative approximations (Hernandez et al., 2014)

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1+\frac{\ln(1-\gamma)}{\lambda}\right)$$

$$F^{-1}\left(1+\frac{\ln(1-\gamma)}{\lambda}\right) = \left(\frac{\lambda\ln(1-\gamma^*)}{\ln(1-\gamma)}\right)^{\kappa} F^{-1}(1+\ln(1-\gamma^*))\,\frac{L_U\left(\frac{\lambda}{-\ln(1-\gamma)}\right)}{L_U\left(\frac{1}{-\ln(1-\gamma^*)}\right)}$$

$$\theta \approx \left(\frac{\lambda\ln(1-\gamma^*)}{\ln(1-\gamma)}\right)^{\kappa},\ \gamma \to 0$$

i.e. $G^{-1}(1-\gamma) \approx \left(\frac{\lambda\ln(1-\gamma^*)}{\ln(1-\gamma)}\right)^{\kappa} F^{-1}(1+\ln(1-\gamma^*))$

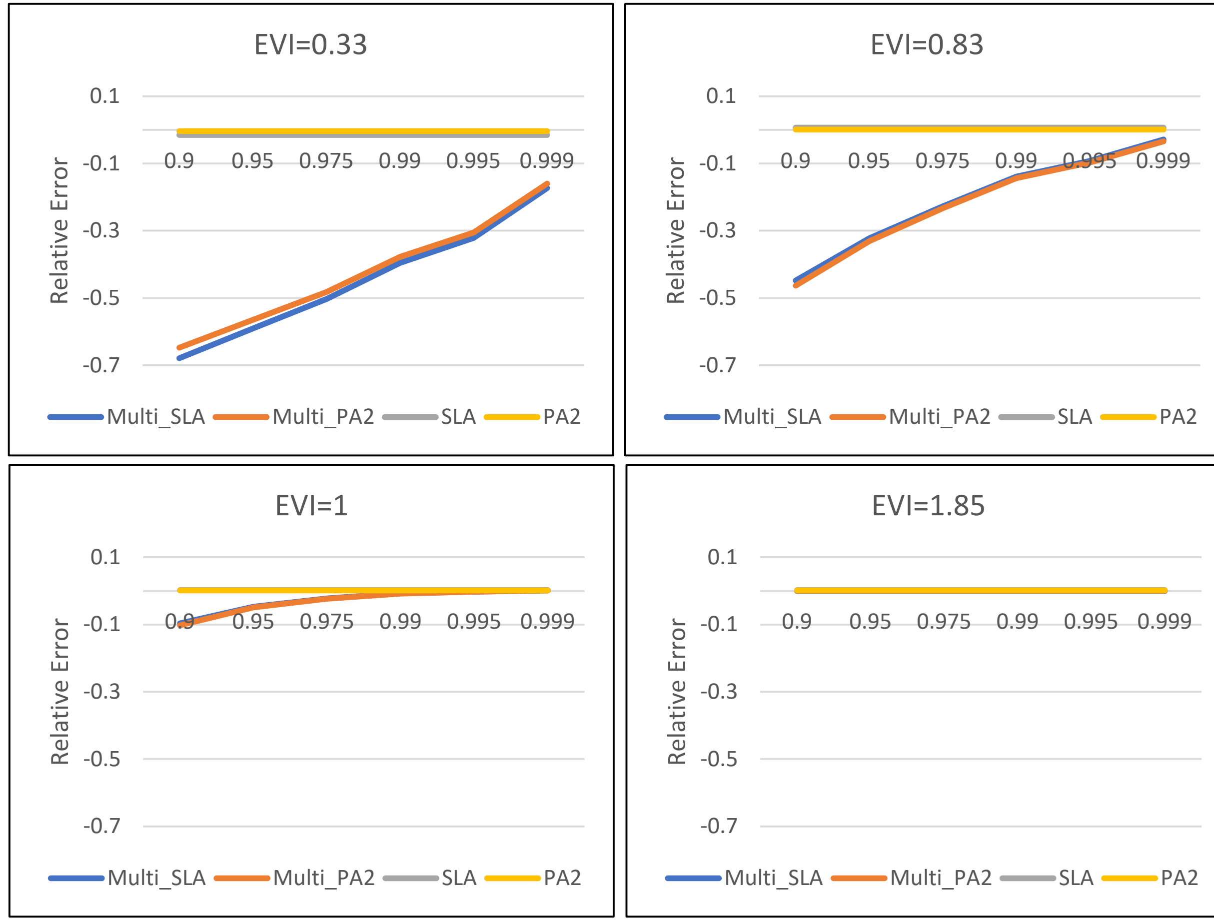
Approximating Quantile: Multiplier

$$RE = \frac{|V - MedAT|}{MedAT}$$

Burr distribution

$$\gamma^* = 0.1, 0.05, 0.025, 0.01, 0.005, 0.001$$

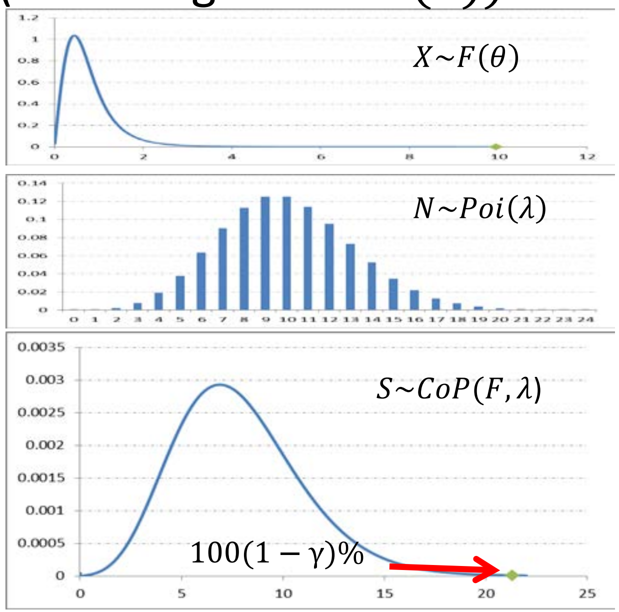
$$\gamma = 0.001$$

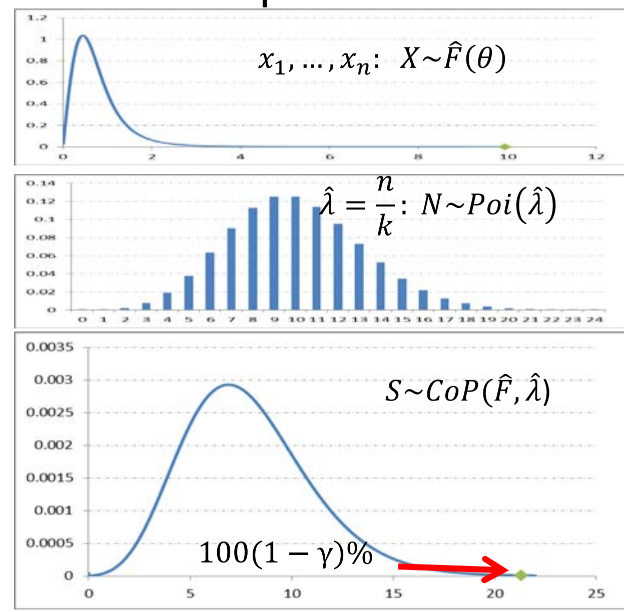


Methodology for evaluation of approximations

Approximate vs. Estimate (using sample)

(assuming N and $F(x)$) | 1000 samples





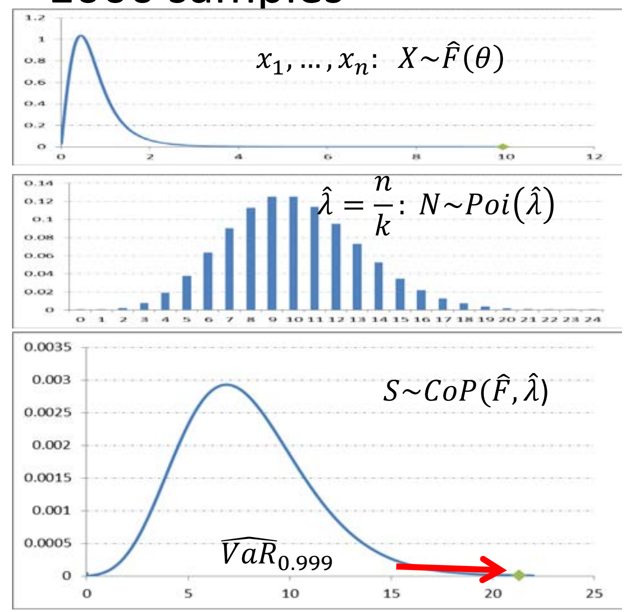
Methodology for evaluation of approximations

Approximate vs. Estimate (using sample)

(assuming N and $F(x)$)

- Approximate the $100(1-\gamma)\%$ quantile using 1 mil MC repetitions.
- Approximately true (AT) $100(1-\gamma)\%$ quantile
- Repeat 1000 times
- Denote the median of the 1000 AT values by MedAT.

1000 samples



Methodology for evaluation of approximations

Approximate (assuming N and $F(x)$)

- Approximate the $100(1-\gamma)\%$ quantile using 1 mil MC repetitions.
- Approximately true (AT) $100(1-\gamma)\%$ quantile
- Repeat 1000 times
- Denote the median of the 1000 AT values by MedAT.

vs.

Estimate (using sample) 1000 samples

- Estimate the $100(1-\gamma)\%$ quantile of the 1000 samples using different methods:
- Parametric : fit Burr to all sample (MLE)
  - MC
  - SLA
  - MP
- Semi-parametric:
  - MC
  - MP

Methodology for evaluation of approximations

Burr Type XII- CDF

$$Burr(x;\eta,\tau,\alpha) = 1 - \left(1 + \left(\frac{x}{\eta}\right)^{\tau}\right)^{-\alpha}, \qquad x > 0$$

$$EVI = \kappa = \frac{1}{\alpha\tau}, \quad E[X] < \infty \; if \; EVI < 1$$

Methodology for evaluation of approximation methods

For each λ, Burr parameter set and each sample;

- *Burr fit*; fit Burr to all sample (MLE)
  - MC
  - SLA
  - MP ($\gamma^* = 0.1, 0.05, 0.025, 0.01$)
- *Empirical*
  - MC
  - MP: κ - Burr ($\gamma^* = 0.1, 0.05, 0.025, 0.01$)

Methodology for evaluation of approximation methods

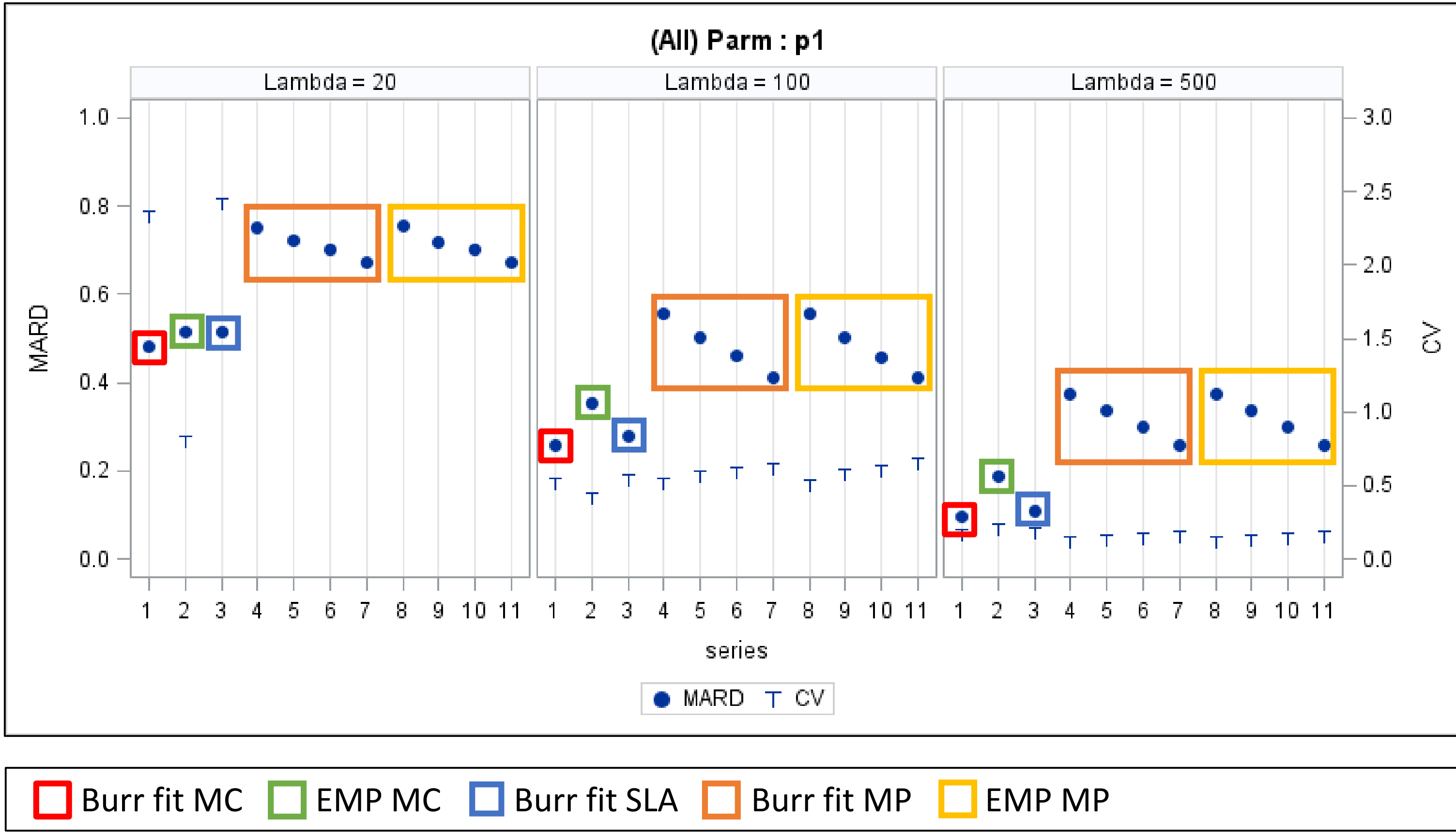
Compare the performance of the approximation techniques relative to the MC distribution:

- To express the quality of the methods we used their median absolute relative deviation (MARD) from MedAT

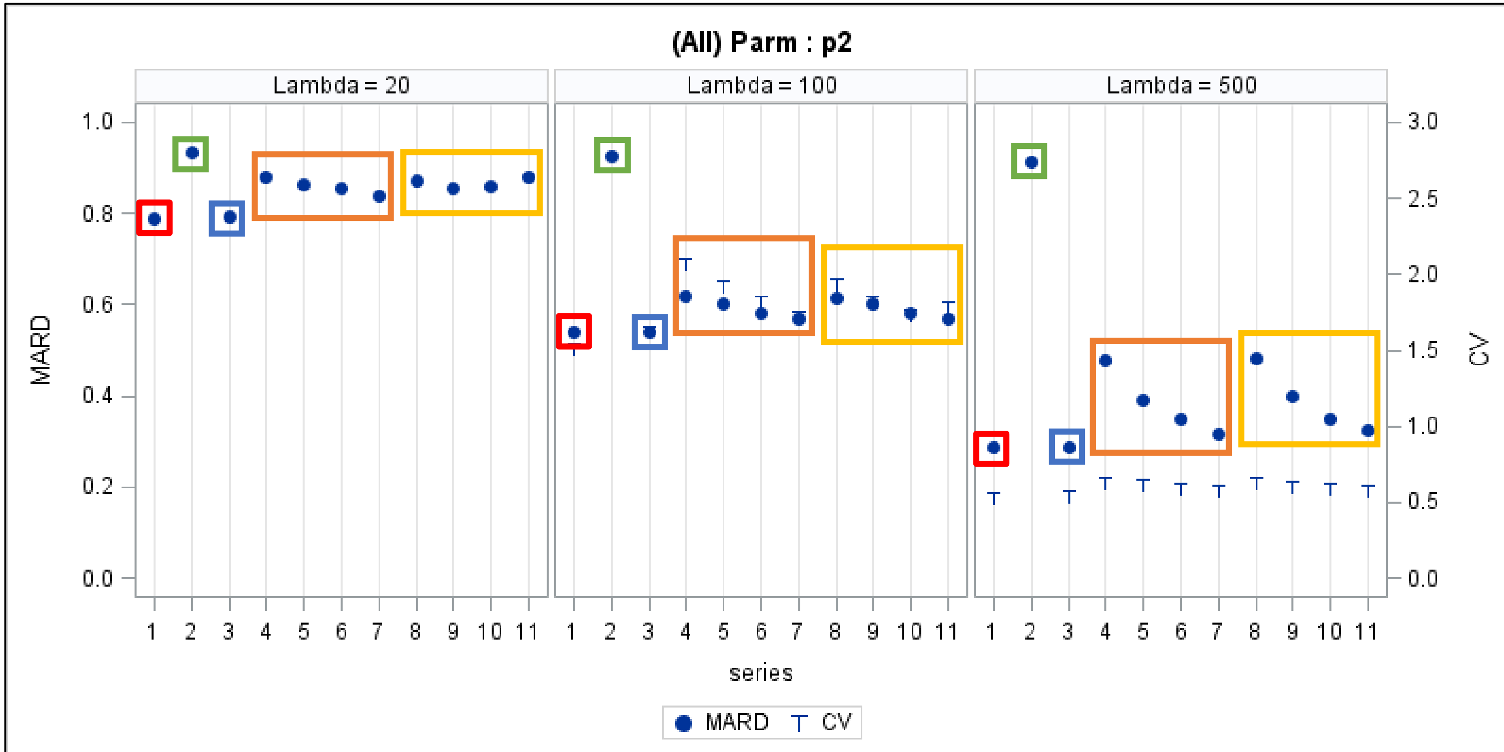
$$MARD(SLA) = Median\left\{\left|\frac{SLA_j}{MedAT} - 1\right|, j = 1,2,\ldots,1000\right\}$$

- $Coefficient\ of\ variance = Var^{0.5}/Mean,$ the variance of the method divided by MC variance

Results - Tail heaviness: Low

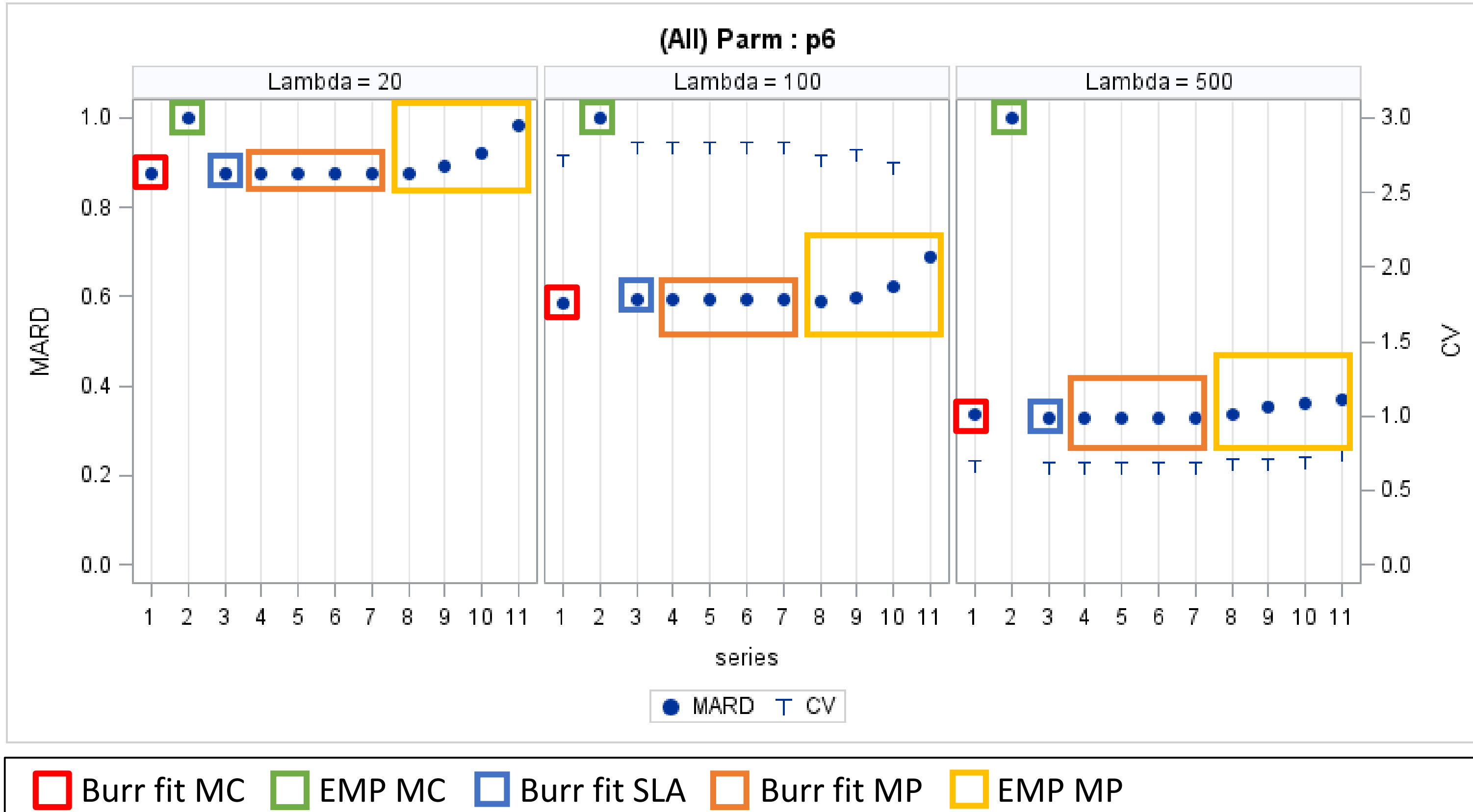


Results - Tail heaviness: Intermediate



Burr fit MC | EMP MC | Burr fit SLA | Burr fit MP | EMP MP

Results - Tail heaviness: High



Concluding Results

	Tail heaviness		
Frequency	Low	Intermediate	High
Low	Bias: MC/SLA Var: EMP	Bias: MC/SLA Var: ~	Bias: MC/SLA/Multi Var: ~
Intermediate	Bias: MC/SLA Var: EMP	Bias: MC/SLA Var: MC/SLA	Bias: MC/SLA/Multi Var: ~
High	Bias: MC/SLA Var: Multi	Bias: MC/SLA/Multi Var: MC/SLA/Multi	Bias: MC/SLA/Multi Var: All

Semi-parametric: Multiplier

If we let $\gamma^* = \frac{k+1}{n+1}$ we can estimate $F^{-1}(1-\gamma^*) = F^{-1}\left(\frac{n-k}{n+1}\right) = X_{n-k,n}$ the $(n-k)$-th order statistic.

We then have an estimator of the following :

$$G^{-1}(1-\gamma) \approx \left(\frac{\lambda}{\gamma}\frac{k+1}{n+1}\right)^{\kappa} X_{n-k,n}$$

Of the same form as introduced in Weissman (1978).

Second order EVT: Multiplier

The second order parameter ρ, rules the rate of convergence in the first order condition, and is the non-positive parameter appearing in the limiting relation

$$\lim_{t\to\infty} \frac{\log L_U(tx) - \log L_U(t)}{d(t)} = h_\rho(t)$$

with $h_\rho(t) = \frac{t^{\rho}-1}{\rho}$ and we assume to hold for all $x > 0$ and where $|d(t)|$ must then be of regular variation with index ρ with $\rho < 0$.

For the tail quantile function this implies

$$U(xt) = U(t)x^{\kappa}\left(1 + h_\rho(x)d(t) + o\big(d(t)\big)\right)$$

Second order EVT: Multiplier

If we take $tx = \frac{\lambda}{\gamma}$ and $t = \frac{1}{\gamma^*} \Rightarrow x = \frac{\lambda\gamma^*}{\gamma}$, then:

$$F^{-1}\left(1 - \frac{\gamma}{\lambda}\right) = U\left(\frac{\lambda}{\gamma}\right) \approx U\left(\frac{1}{\gamma^*}\right)\left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}\left(1 + h_\rho\left(\frac{\lambda\gamma^*}{\gamma}\right) d\left(\frac{1}{\gamma^*}\right)\right)$$

$$= F(1 - \gamma^*)\left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}\left(1 + h_\rho\left(\frac{\lambda\gamma^*}{\gamma}\right) d\left(\frac{1}{\gamma^*}\right)\right)$$

i.e. $G^{-1}(1-\gamma) \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}\left(1 + h_\rho\left(\frac{\lambda\gamma^*}{\gamma}\right) d\left(\frac{1}{\gamma^*}\right)\right) F^{-1}(1-\gamma^*)$

Conclusion and future research

- The multiplier approach is comparable to other methods in intermediate tail heaviness with higher frequency.
- Provides semi-parametric estimation possibilities

- Next step to do simulation study

Thank you