A multiplier approach for approximating and estimating extreme quantiles of compound frequency distributions using the less extreme quantiles of the severity distribution

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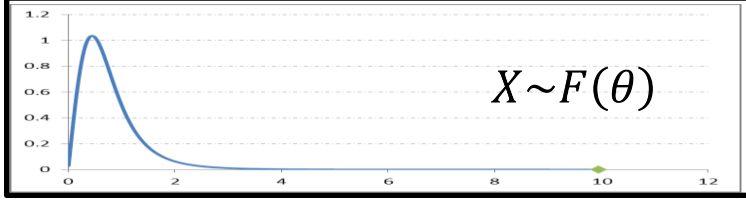
Compound distributions and quantile estimation

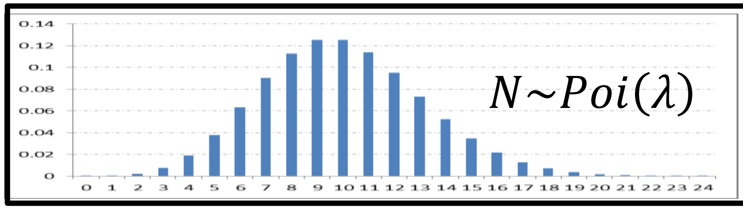
- **Many applications and particularly in insurance and risk management.**
- **Example:**
    - Insurance; model the distribution of aggregate claims of an insurance policy over a fixed period or,
    - Operational risk; model the annual aggregate loss distribution.
    - Credit risk; model the annual aggregate loss distribution.

Theoretical framework



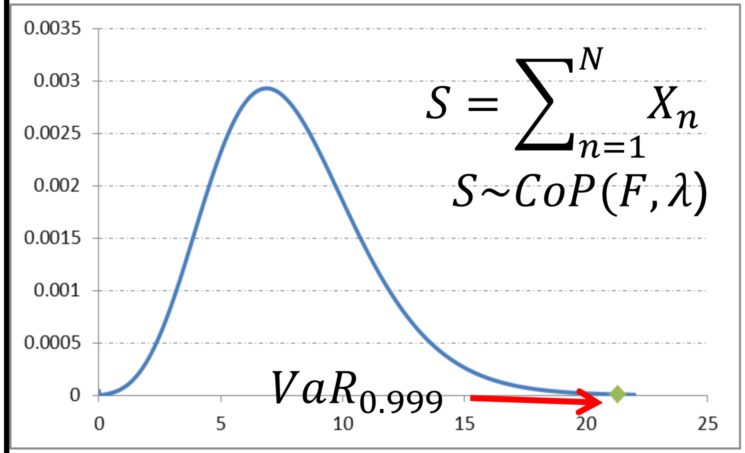
Let $X_1, \dots, X_N \sim F$

Loss severity distribution



$N \sim Poi(\lambda)$

Frequency distribution



$S = \sum_{n=1}^{N} X_n \sim CoP(F,\lambda) = G$

Aggregate loss distribution

We are interested in estimating the 99.9% quantile (VaR) of the Annual Aggregate Loss Distribution, i.e. the aggregate loss that is expected to be exceeded once in 1000 years.

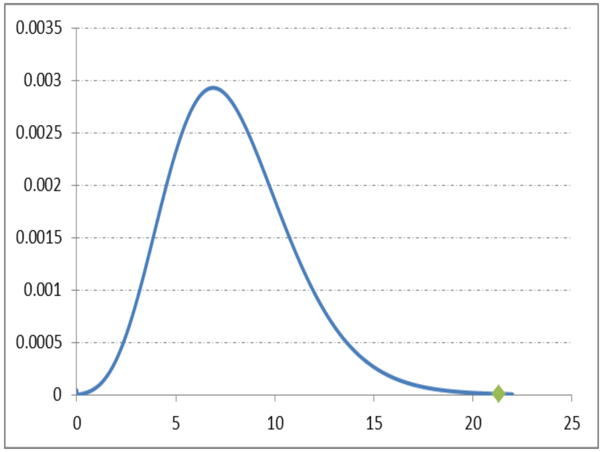
Approximation methods

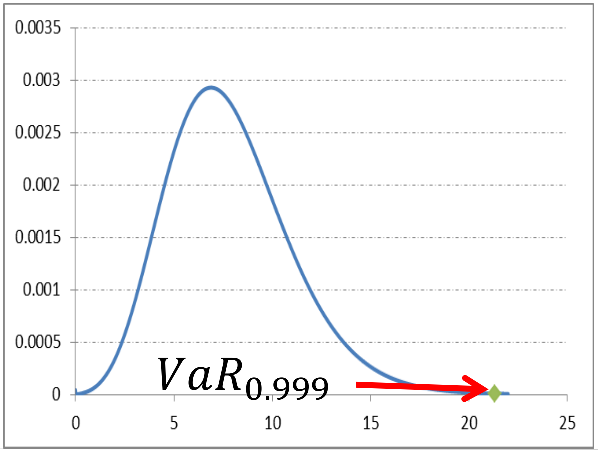
Approximating the distribution

1) Monte Carlo (MC) approach
2) Numerical approximation alternatives:
    - Panjer recursion (Panjer, 1981)
    - Fast Fourier transforms (Meyers,1983)

Approximating the quantile (closed-form)

1) Single loss approximations
2) Perturbative approaches

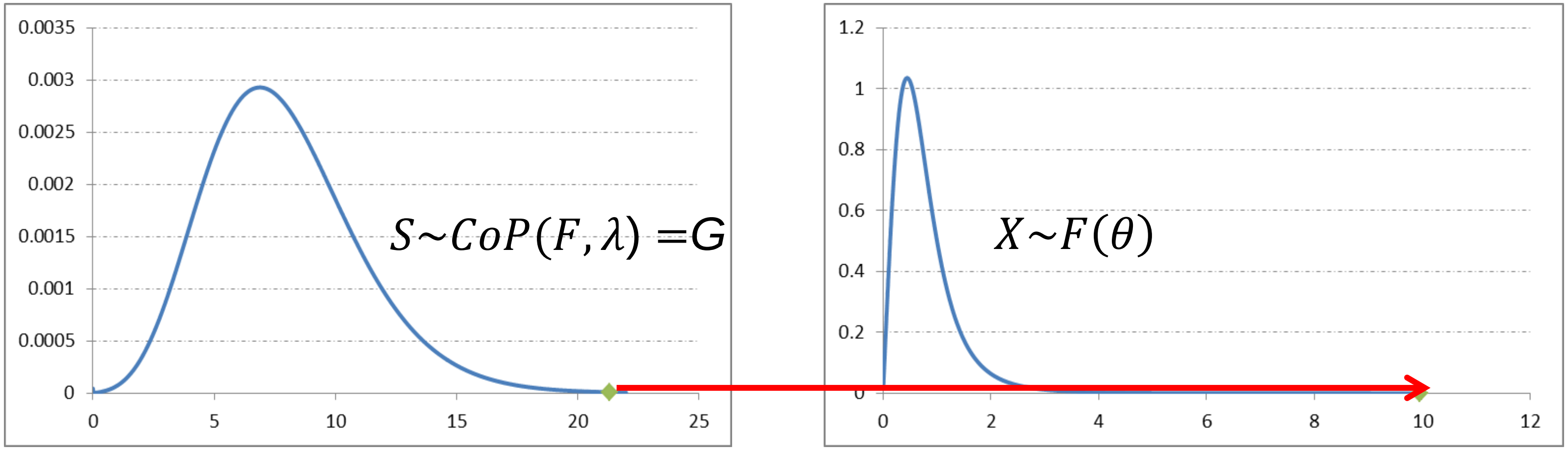




Approximating Quantile: Closed form

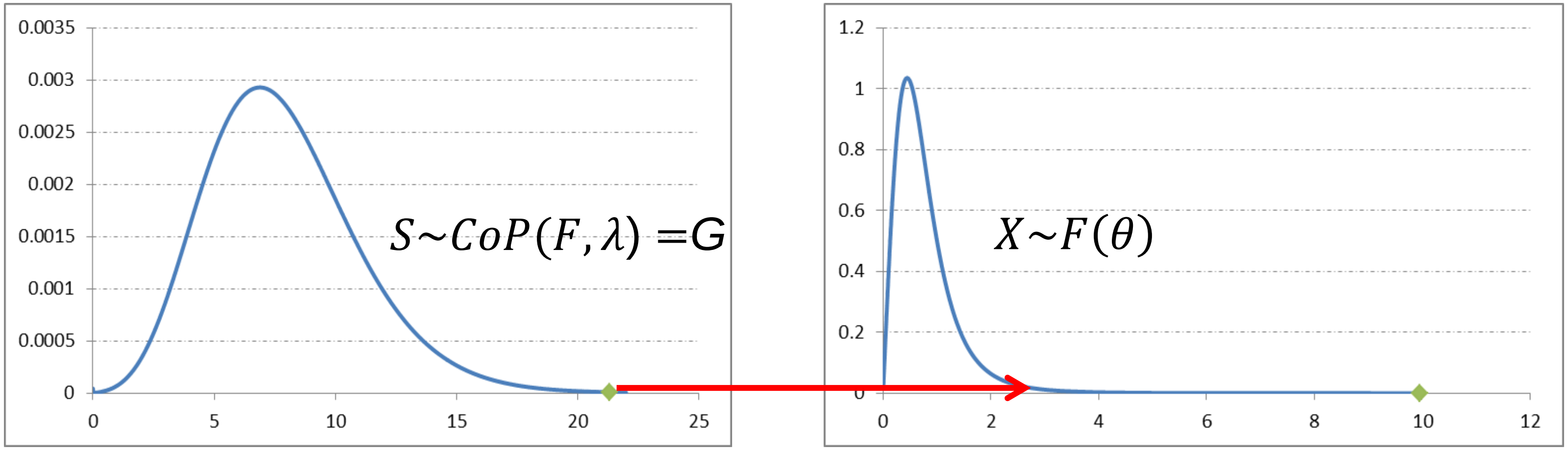
Single Loss Approximations - Böcker and Klüppelberg (2005) :

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right)$$



Theoretical framework

Can we approximate the $100(1-\gamma)\%$ quantile of the compound distibution G by using a multiplier (denoted by θ) and the $100(1-\gamma^*)\%$ quantile the distribution of F where $\gamma^* > \gamma$?

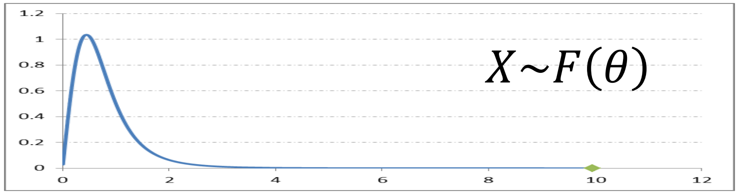


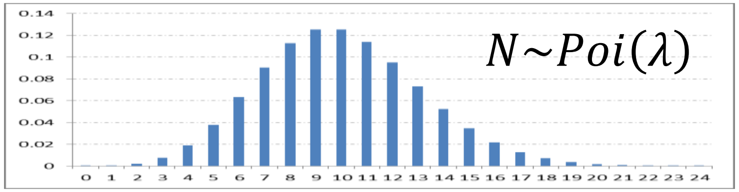
$$G^{-1}(1-\gamma) \approx \theta \times F^{-1}(1-\gamma^*)\ ;\ \theta \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}, \kappa = EVI$$

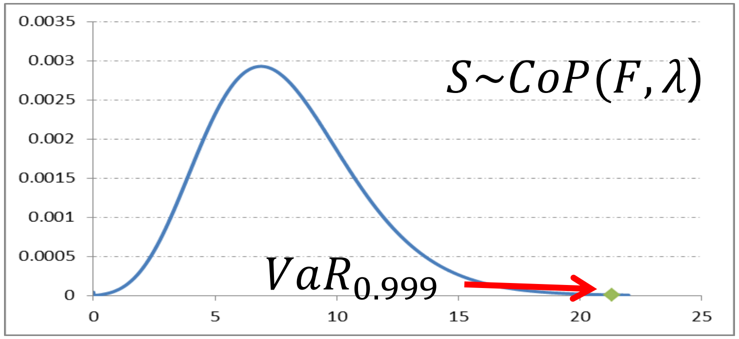
Methodology for evaluation of approximations

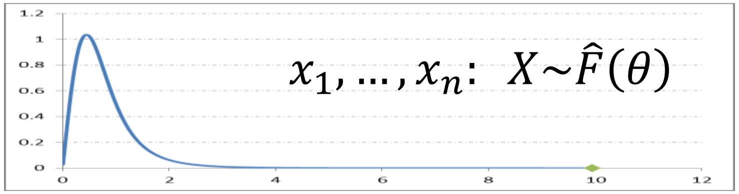
Approximate vs. Estimate (using sample)

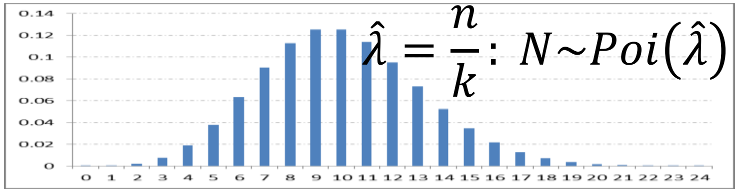
(assuming N and $F(x)$) 1000 samples

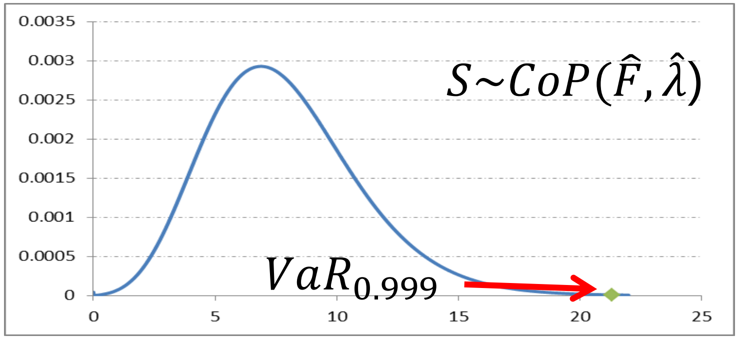








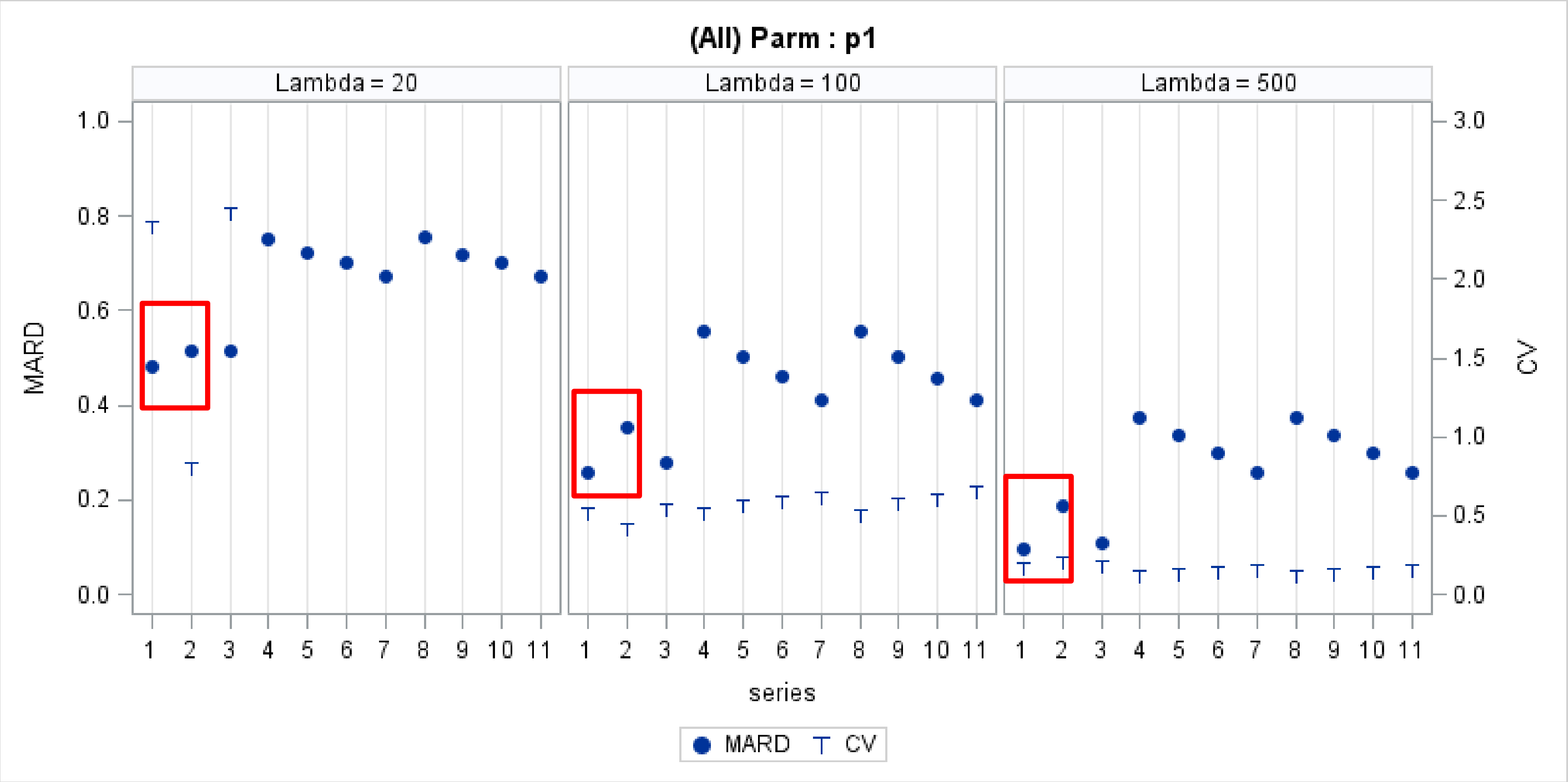




Concluding Results

	Tail heaviness		
Frequency	Low	Intermediate	High
Low	Bias: Var:		
Intermediate			
High			

Results - Tail heaviness: Low



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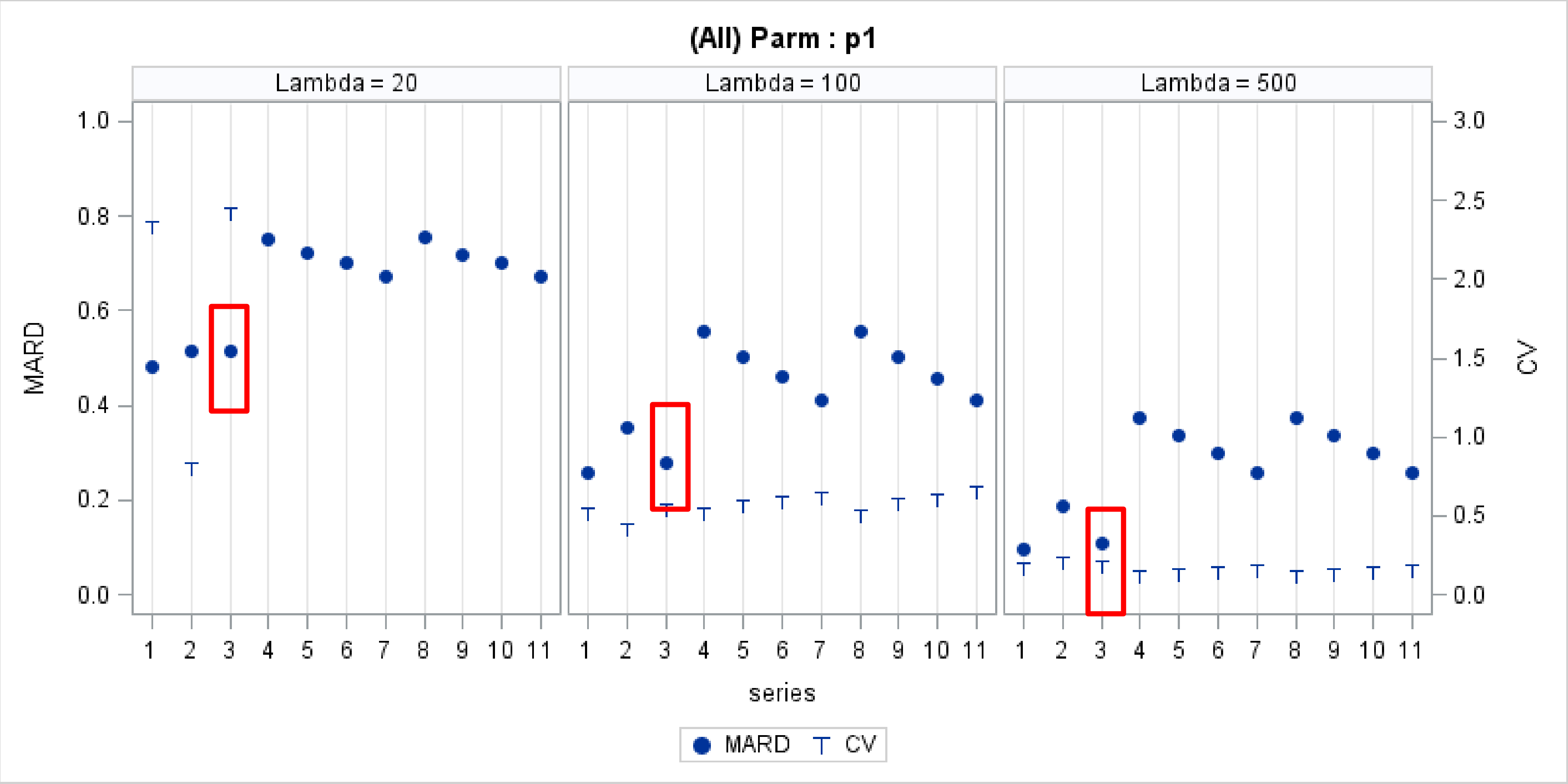

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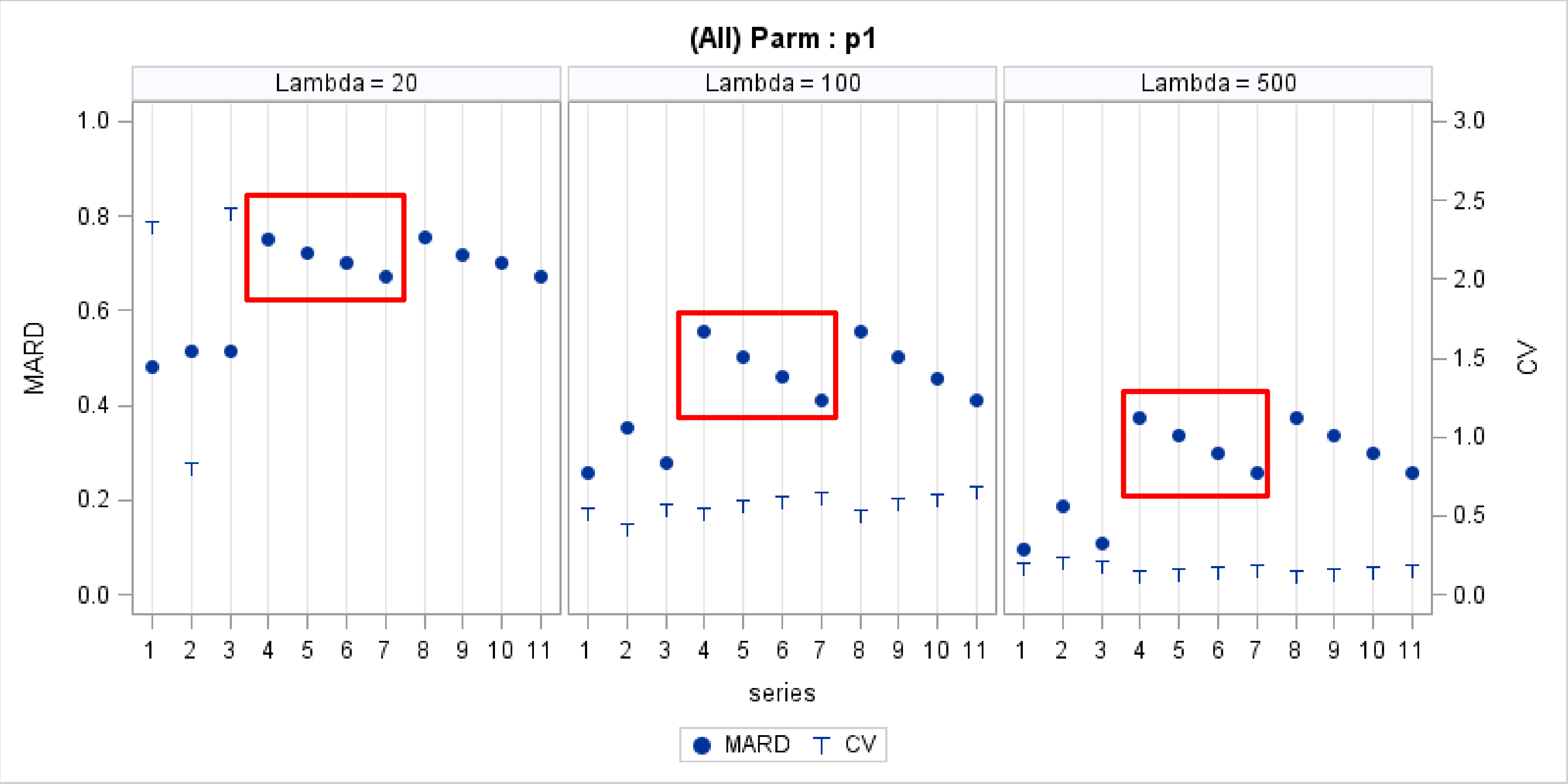
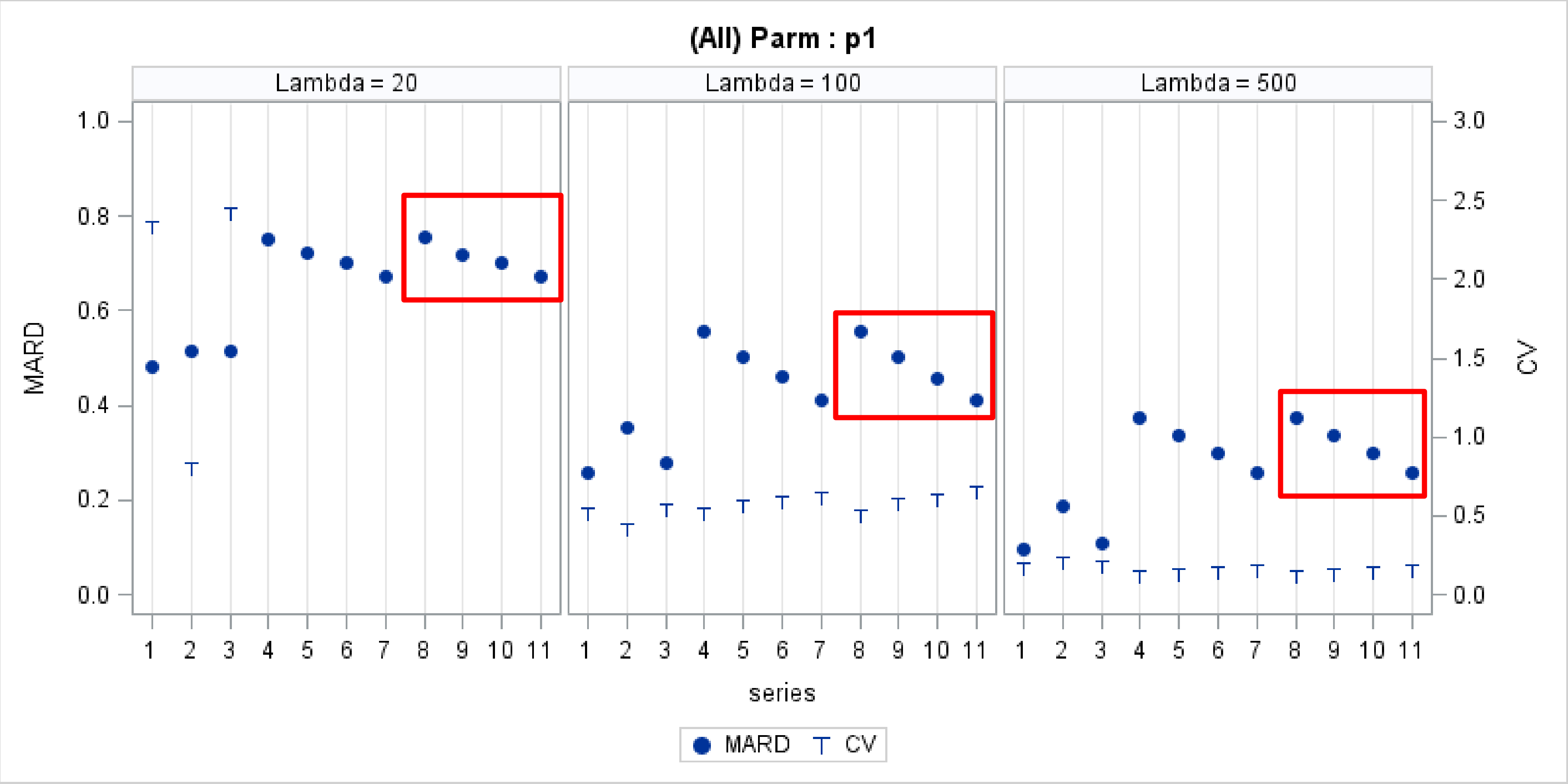
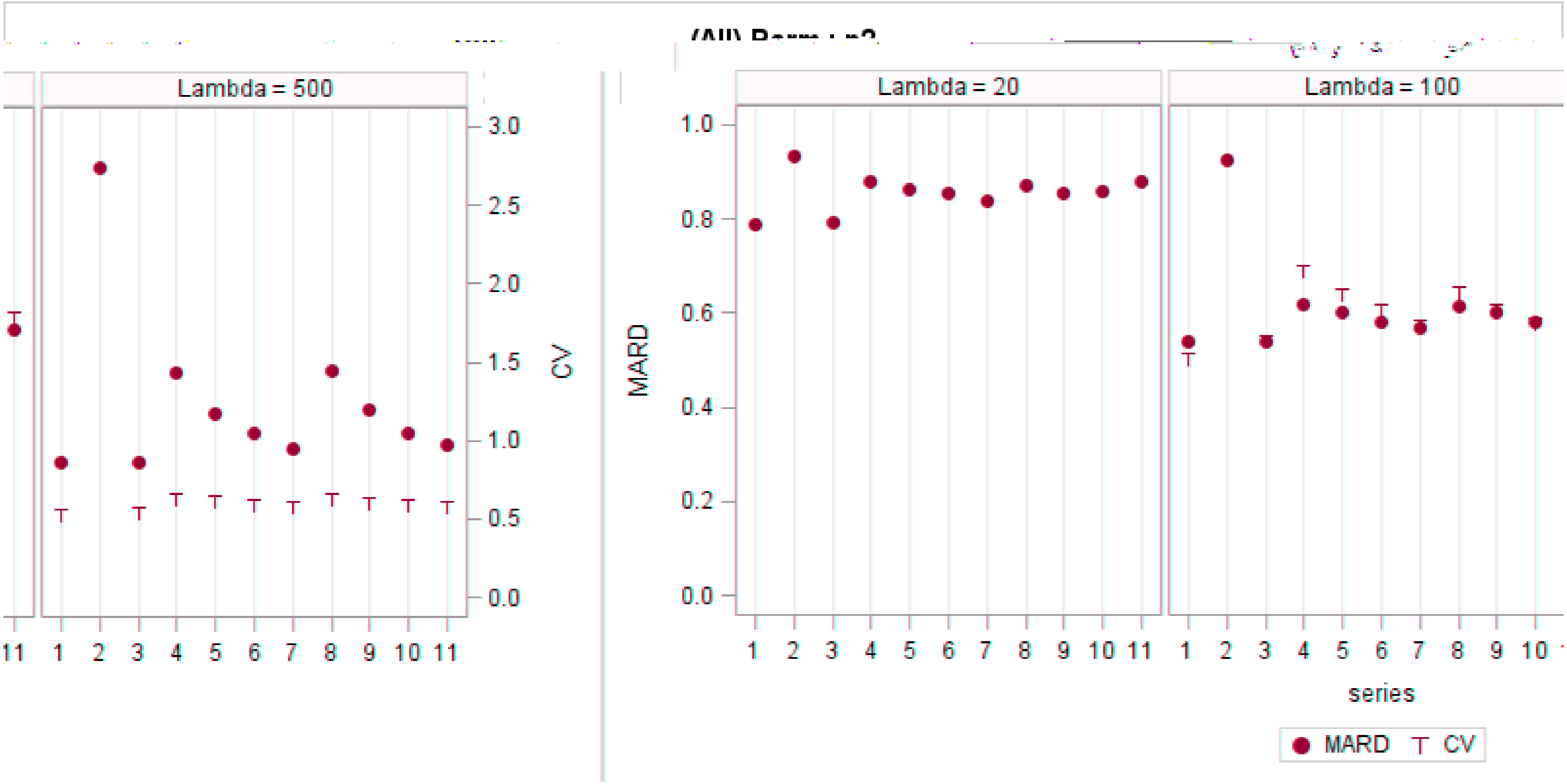


Results - Tail heaviness: Low



Results - Tail heaviness: Intermediate



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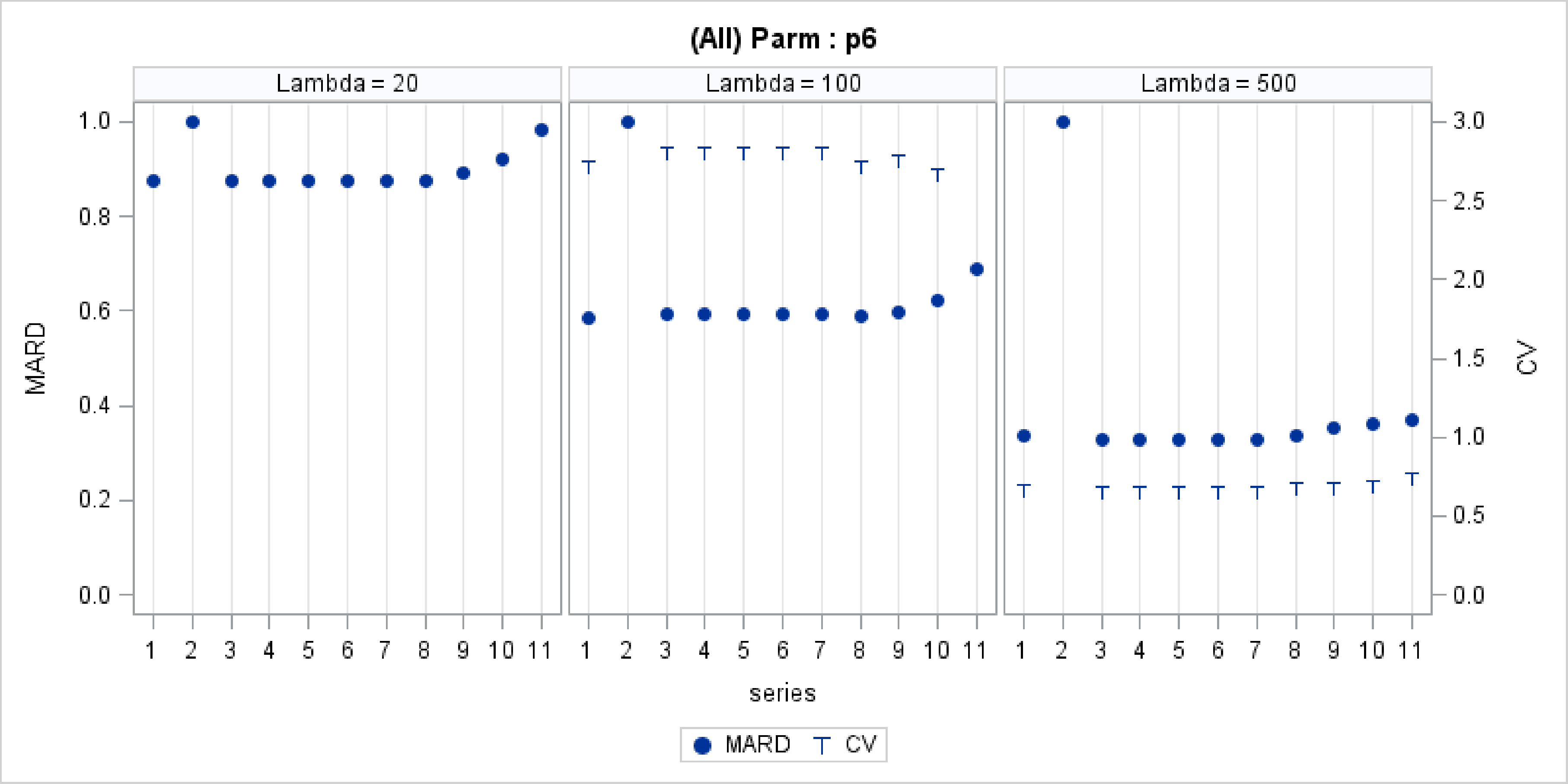
Results - Tail heaviness: High



Concluding Results

	Tail heaviness		
Frequency	Low	Intermediate	High
Low	Bias: MC/SLA Var: EMP	Bias: MC/SLA Var: ~	Bias: MC/SLA/Multi Var: ~
Intermediate	Bias: MC/SLA Var: EMP	Bias: MC/SLA Var: MC/SLA	Bias: MC/SLA/Multi Var: ~
High	Bias: MC/SLA Var: Multi	Bias: MC/SLA/Multi Var: MC/SLA/Multi	Bias: MC/SLA/Multi Var: All

Conclusion and future research

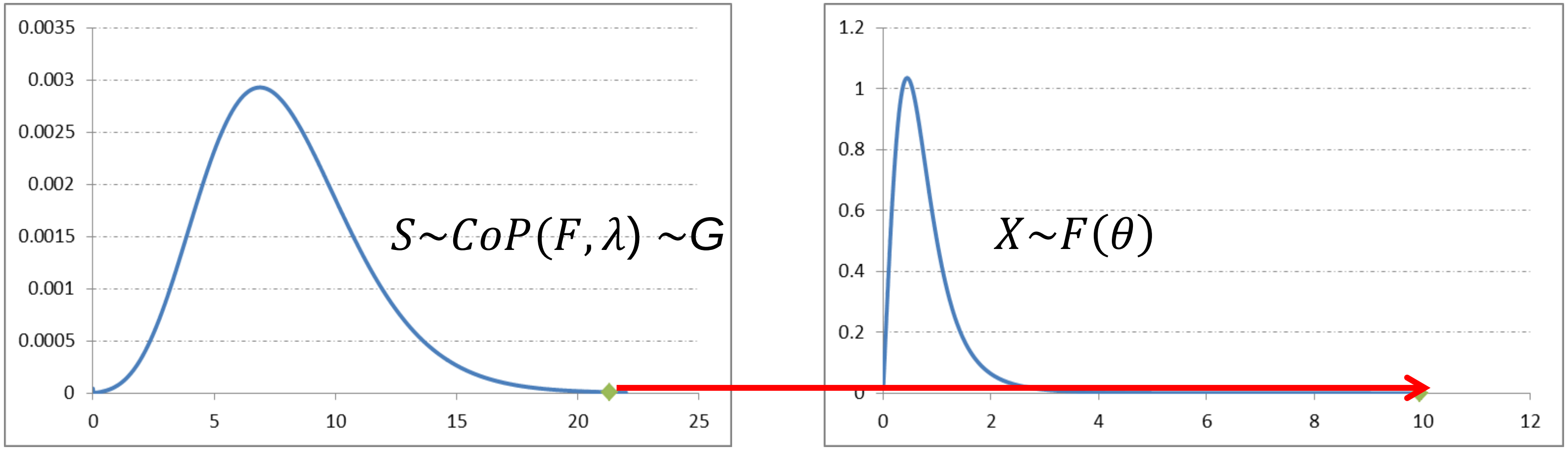
- The multiplier approach is comparable to other methods in intermediate tail heaviness with higher frequency.
- Provides non-parametric estimation possibilities

- Improve multiplier by using second order approximations
- Improve methods for tail index approximation

Approximating Quantile: Closed form

Single Loss Approximations - Böcker and Klüppelberg (2005) :

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right)$$



Approximating Quantile: Closed form

SLAD: Degen (2010), using second order exponentiality, derived an improved single loss approximation

- *finite mean* ($\mu = E(X) < \infty$)

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right) + \lambda\mu,$$

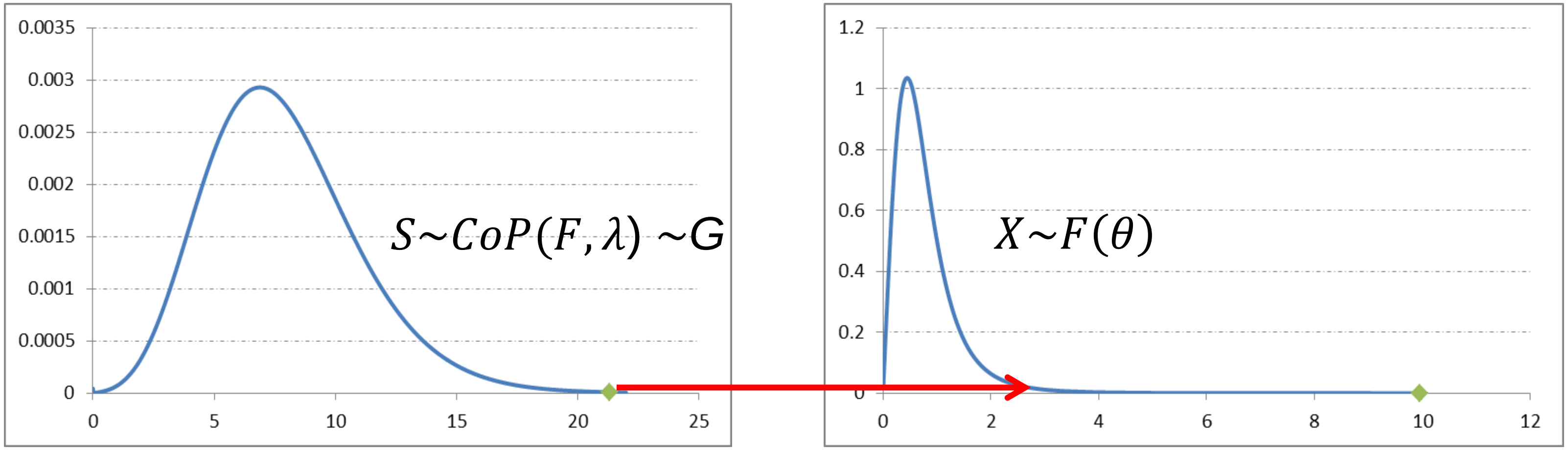
- *infinite mean models* ($\mu = E(X) = \infty$)

$$G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right) + \gamma F^{-1}\left(1-\frac{\gamma}{\lambda}\right)\frac{C_\kappa}{1-\frac{1}{\kappa}}$$

where $C_\kappa = (1-\kappa)\frac{\Gamma^2\left(1-\frac{1}{\kappa}\right)}{2\Gamma\left(1-\frac{2}{\kappa}\right)}$, Γ the gamma function, $\kappa = \mathrm{EVI}$

Approximating Quantile: Multiplier

Can we approximate the $100(1-\gamma)\%$ quantile of the compound distibution G by using a multiplier (denoted by θ) and the $100(1-\gamma^*)\%$ quantile the distribution of F where $\gamma^* > \gamma$?



$$G^{-1}(1-\gamma) \approx \theta \times F^{-1}(1-\gamma^*)$$

Extreme value index / sub-exponential

Note that the sub-exponential class includes all regularly varying densities.

A positive measurable function h is regularly varying with parameter β $(h \in RV_\beta)$, if

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\beta \text{ for all } x > 0.$$

Extreme value index / sub-exponential

In the case of a probability density f by Karamata’s Theorem, if $f \in RV_{-\frac{1}{\kappa}-1}$ then $\bar{F} \in RV_{-1/\kappa}$ where $\bar{F}(x) = 1 - F(x)$ (see e.g. Embrechts et al. 1997).

In terms of its tail quantile function $U(t) = F^{-1}\left(1 - \frac{1}{t}\right)$, this is equivalent to $U \in RV_{\kappa}$.

κ is the extreme value or tail index of the distribution.

Extreme value index / sub-exponential

Regularly varying functions are functions which can be represented by power functions multiplied by slow varying functions, i.e. if $\lim_{t\to\infty} L(tx)/L(t) = 1$ then $h(x) = x^{\beta}L(x)$ where $L(x)$ is a slow varying function.

We write L_F and L_U for the slow varying function associated with $\bar{F}$ and U respectively.

Approximating Quantile: Multiplier

F is regularly varying then, under certain limiting conditions, $G^{-1}(1-\gamma) \approx F^{-1}\left(1-\frac{\gamma}{\lambda}\right)$ (SLA, see e.g. Bocker and Kluppelberg 2005 or Degen 2010)

$$\theta = \frac{G^{-1}(1-\gamma)}{F^{-1}(1-\gamma^*)} \approx \frac{F^{-1}\left(1-\frac{\gamma}{\lambda}\right)}{F^{-1}(1-\gamma^*)} = \frac{U\left(\frac{\lambda}{\gamma}\right)}{U\left(\frac{1}{\gamma^*}\right)} = \frac{\left(\frac{\lambda}{\gamma}\right)^{\kappa} L_U\left(\frac{\lambda}{\gamma}\right)}{\left(\frac{1}{\gamma^*}\right)^{\kappa} L_U\left(\frac{1}{\gamma^*}\right)}$$

$$\theta \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\kappa}, \qquad \gamma \to 0$$

Approximating Quantile: Multiplier

Corrections following Degen (2010)

- *finite mean* ($\mu = E(X) < \infty$)

$$G^{-1}(1-\gamma) \approx \theta F^{-1}(1-\gamma^{*}) + \mu\lambda$$

- *infinite mean models* ($\mu = E(X) = \infty$)

$$G^{-1}(1-\gamma) \approx \theta F^{-1}(1-\gamma^{*}) \left(1 + \gamma \frac{c_{\kappa}}{1-\frac{1}{\kappa}}\right)$$

where $C_{\kappa} = (1-\kappa)\frac{\Gamma^{2}\left(1-\frac{1}{\kappa}\right)}{2\Gamma\left(1-\frac{2}{\kappa}\right)}$, Γ the gamma function

Approximating Quantile: Multiplier

If $\kappa = 0$, one may interpret the tail index κ as the ultimate slope in the log–log plot of U, i.e. writing $U(t) = e^{\varphi(log(t))}$ one has $\varphi'(t) \to \kappa$, as $t \to \infty$ (see, for example, Degen and Embrechts (2008)).

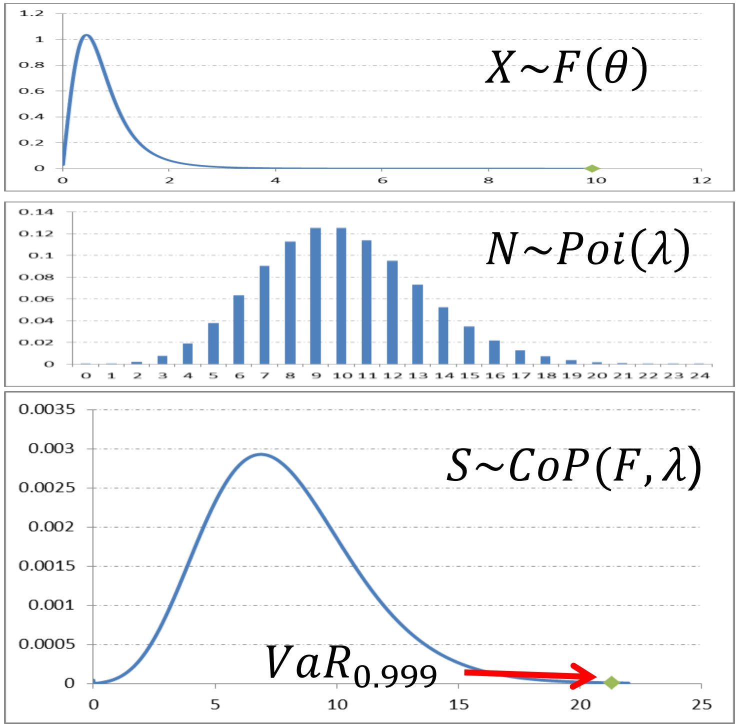
Therefore we estimate $\theta \approx \left(\frac{\lambda\gamma^*}{\gamma}\right)^{\varphi'(\log(t))}$

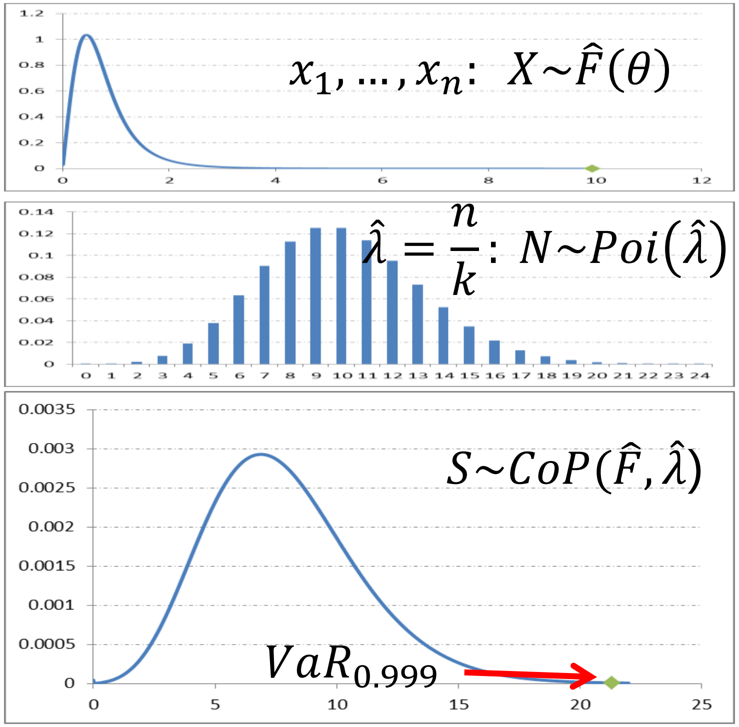
Methodology for evaluation of approximations

Approximate vs. Estimate (using sample)

(assuming N and $F(x)$)

1000 samples





Methodology for evaluation of approximations

Monte Carlo (MC) study

Assumed a Poisson frequency: $\lambda = 10, 20, 50, 100, 200, 500$

Severity distributions – Burr Type XII (6 parameter sets)

$EVI = 0.33, 0.83, 1, 1.33, 1.85, 2.35$

Quantiles : $\gamma = 0.05, 0.025, 0.01, 0.005, 0.001$

$\gamma^* = 0.1, 0.05, 0.025, 0.01$

Methodology for evaluation of approximations

Burr Type XII- CDF

$$Burr(x;\eta,\tau,\alpha)=1-\left(1+\left(\frac{x}{\eta}\right)^{\tau}\right)^{-\alpha}, \qquad x>0$$

$$EVI=\kappa=\frac{1}{\alpha\tau}, \quad E[X]<\infty \; if \; EVI<1$$

Methodology for evaluation of approximation methods

For each λ, Burr parameter set and each sample;

- *Burr fit*; fit Burr to all sample (MLE)
  - MC
  - SLA
  - MP ($\gamma^* = 0.1, 0.05, 0.025, 0.01$)
- *Empirical*
  - MC
  - MP: κ - Burr ($\gamma^* = 0.1, 0.05, 0.025, 0.01$)

Methodology for evaluation of approximation methods

Compare the performance of the approximation techniques relative to the MC distribution:

- To express the quality of the methods we used their median absolute relative deviation (MARD) from MedAT

$$MARD(SLA) = Median\left\{\left|\frac{SLA_j}{MedAT} - 1\right|, j = 1,2,\ldots,1000\right\}$$

- $Coefficient\ of\ variance = Var^{0.5}/Mean$, the variance of the method divided by MC variance

Conclusion and future research

- The multiplier approach is comparable to other methods in intermediate tail heaviness with higher frequency.
- Provides non-parametric estimation possibilities

- Improve multiplier by using second order approximations
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Thank you