

# Pricing variable annuity guarantees in South Africa under a Variance-Gamma model

By AM Ngugi, E Maré and R Kufakunesu

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## ABSTRACT

The purpose of this study is to investigate the pricing of variable annuity embedded derivatives using a suitably refined model for the underlying assets, in this case the Johannesburg Securities Exchange FTSE/JSE All Share Index (ALSI). This is a practical issue that life insurers face worldwide in the management of embedded derivatives. We consider the Variance-Gamma (VG) framework to model the underlying data series. The VG process is useful in option pricing given its ability to model higher moments, skewness and kurtosis and to capture observed market dynamics. The framework is able to address the inadequacies of some deterministic pricing approaches used by life insurers, given the increasing complexity of the option-like products sold.

## KEYWORDS

Embedded derivatives; options; variable annuity; Variance-Gamma; hedging

## CONTACT DETAILS

Eben Maré, Department of Mathematics and Applied Mathematics, University of Pretoria, Private Bag X20, Hatfield, 0028; Tel: +27(0)12 420 2835; Email: [eben.mare@up.ac.za](mailto:eben.mare@up.ac.za)

## 1. INTRODUCTION

### 1.1 Background

1.1.1 The financial industry has, in the past century, been characterised by periods of market stability such as the early 1960s and early 1980s followed by spells of instability, with market declines, as in the late 1980s, for example. The early 1990s were also relatively

stable years in the markets with the latter part of the decade then experiencing significant volatility. Other market variables such as interest rates and exchange rates have also shown high levels of volatility. Financial solutions have been sought for these market vagaries (see, for example, Zenios, 1995, and Cummins & Weiss, 2009) with financial hedging widely used to protect market participants.

1.1.2 In the same period, the pool of market participants has grown with different industries seeking solutions from the capital markets. This has taken place through various initiatives such as raising finance, moving from traditional share capital as a source of finance and creating products that protect market participants against the impacts of adverse exposure. The insurance industry has been one such market participant.

1.1.3 The last three decades have brought substantial changes to the nature of operational methods of the insurance industry. In this period, insurers have moved to combine their models of mortality with finance models to build a framework that prices their products fairly while ensuring sound risk management. In the process, the financial and actuarial worlds have become more interconnected than ever (Embrechts, 2000).

1.1.4 Business competitiveness has provoked insurers to move from traditional assurance products to offerings that combine insurance and savings elements. In the same period, policyholders have become more sophisticated in their needs. This, coupled with the advent of powerful computational facilities, has led to increasingly complex products.

1.1.5 These complex products have necessitated a new approach to managing risk. Hedging is defined by Joshi (2003) as an attempt to counteract or, where possible, remove the risks that an entity faces by holding an offsetting position. Some life insurance companies have restructured themselves by creating a directed treasury to conduct their asset-liability hedging activities. These companies now typically have an asset management function and an intermediary that links the insurer's liabilities with its hedging assets.

1.1.6 The central theme of this study is the variable annuity product and the different structural presentations of the product to the clients taking out insurance. In particular, embedded derivatives in variable annuities are considered along with the techniques for pricing them in the context of the increasing complexity and competitive importance of these guarantees in the life insurance industry.

1.1.7 A variable annuity is an annuity in which the premiums are invested mainly in the financial markets and receipts by the annuitant are dependent, in some way, on the performance of the underlying assets. The need for differentiation in the life insurance industry has meant that insurers will normally provide guarantees of some form to the annuitant. There are four main guarantees in variable annuities: Guaranteed Minimum Maturity Benefit (GMMB), Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB) and Guaranteed Minimum Withdrawal Benefit (GMWB).

1.1.8 The GMDB benefit guarantees a lump-sum on death regardless of the performance of the underlying assets. The GMMB provides a minimum guarantee to the policyholder at maturity. The GMIB guarantees income until the policyholder dies. The GMAB offers a guaranteed amount regardless of the performance of the underlying assets,

differing from the GMMB in that for the latter, the guarantee is only applicable at maturity. These products are normally written collectively as GMxB (Olivieri & Pitacco, 2011). This paper considers the first two, describing how financial engineering can provide solutions to pricing these annuities.

## 1.2 Objectives

1.2.1 This study has three main objectives:

- To provide familiarity on practical issues facing life insurers, which arise from embedded options in GMxB products;
- To implement the Variance-Gamma framework in embedded option pricing as a refined model for movement of the underlying asset;
- To provide a comparison between the Variance-Gamma framework with the more conventional and actuarially accepted regime switching framework (Ngugi, Maré & Kufakunesu, 2014).

1.2.2 Foroughi, Jones & Dardis (2003) discuss the topic of investment guarantees in a South African context but do not venture into model theory. Maitland (2001) discussed the immunisation of nominal liabilities in South Africa and used principal component analysis in his research. Thomson (2011) discusses the pricing of liabilities in an incomplete market with a specific application to the South African retirement fund whereas Raubenheimer & Kruger (2010) propose a dynamic stochastic programming model for use in asset liability modelling (ALM) and note the necessity of further research on this area in South Africa.

1.2.3 The value of this study lies in implementing the non-conventional embedded option pricing models in a South African life insurance setting using the FTSE/JSE All Share index (ALSI), analysing the results obtained and trying to find the best practice for the long-term sustainability of the industry. The authors are not aware of any such study in a South African setting.

## 2. LITERATURE REVIEW

2.1 Consider the situation of Equitable Life, the first mutual company established as the ‘Society for Equitable Assurances on Lives and Survivorships’. The Equitable sold deferred annuities to policyholders with the option of a Guaranteed Annuity Rate (GAR) or the Current Annuity Rate (CAR). The exercise of these options was mostly reliant on the prevailing interest rate and, although CAR was the more generous annuity, market volatility could lead to the GAR option being exercised. The insurer did not hedge itself against the exercise of these guarantees and was exposed to the possibility of heavy losses in the case of adverse market movements.

2.2 In the early 1990s, the CAR fell below the GAR and the option was exercised by the policyholders in large numbers. In a sworn affidavit during his litigation, Christopher Headdon, the CEO of The Equitable at the time, noted that the cost of the guarantees to the company was in excess of £1 billion but, “at no time did Equitable ever hedge or reinsure

adequately against the GAR risk to counteract it” (Glick & Snowden, unpublished: 4). In the Penrose report submitted in 2004, the committee noted that the Equitable Life management team could have taken measures to avoid the failure by instituting a risk management framework to protect it from the guarantees that lay embedded in its product offerings (Penrose, 2004).

2.3 As the Equitable case was waning, in the midst of the 2008 financial crisis, life insurers that were not well prepared were facing a crisis not dissimilar in characteristic. Life insurers in the United States faced substantial losses arising from significant amounts of business written on variable annuities with guarantees. In a report delivered to the Actuarial Society of South Africa (ASSA) following the crisis, the total losses from these annuities was reported as in excess of \$4 billion (approximately R50 billion at the time of writing) as shown in Table 1.

TABLE 1. Variable annuity losses during the 2008 financial crisis

Company	Loss
Manulife	\$1.5 billion
The Hartford	\$834 million
ING	\$700 million
Axa	\$520 million
Old Mutual (Bermuda)	\$500 million
Philadelphia Lincoln Financial	\$145 million

Source: Addae (unpublished)

2.4 These reported losses were not attributable solely to guarantees. Some blame could be laid on poor risk management systems that did not fully account for the complex guarantees in question. These entities had no structures in place to accurately price and subsequently hedge against adverse exposures.

2.5 In the case of Old Mutual-Bermuda (OMB), the insurer sold poorly-priced policies with accumulation and death benefit guarantees. There were weaknesses in the overall risk management framework: guarantees were underpriced and the hedging approach was inadequate. None of these weaknesses were identified despite policy sales growing exponentially. At the time of the global financial crisis, credit spreads widened and global equity markets fell. This resulted in the guarantees maturing, leading in turn to substantial losses and the subsequent need for capital injection. As Old Mutual noted in 2009 and 2010, the demand for products with guarantees continued to grow but the challenge remained raising enough capital to sustain them (Old Mutual, 2010).

2.6 The situation in Japan was no different. Though the demographics of the region favour the sale of variable annuities, after the 2008 crisis, major industry players Hartford Life, ING Life and Mitsui Life decided to discontinue activities in this market. In a report

on the Japanese variable annuity industry, Asada (2009) notes that deficiencies in risk management were a major factor that led to the financial losses arising from the realisation of the guarantees. Though the use of reinsurance helped mitigate the risk in some instances, the author alludes to accurate pricing and hedging as important ideals that could have assisted in the circumstances.

2.7 These studies illustrate but a few of the many insurers that continue to face difficulties in pricing and hedging their guarantees, and form the background to this study. We explore this issue further in the context of the need for business agility, "... the ability to identify, anticipate and respond to relevant changes in operating conditions—those changes that directly impact an insurer's ability to achieve sustained performance" (Capgemini, 2011: 5).

2.8 The issue of how to price for these guarantees has remained a challenge for many life insurance companies. The work done in the early 1980s by David Wilkie (described in Wilkie, Waters & Yang, 2003) on reserving for the guarantees in unit-linked products resulted in insurers shying away from such features. As the derivatives market continued to evolve, the guarantees started re-emerging. Hedging the guarantees has, however, presented itself as an even bigger challenge in the wake of huge market volatility. Wu (2009) notes that the 2008 crisis has made it imperative for insurance companies to be concerned not only about the correct valuation of the embedded options, but also about the containment of risks arising from holding the options through appropriate hedging.

2.9 Statutory and professional bodies have continued to give guidance on the issue from the understanding that such financial guarantees, if not well checked, can lead to the collapse of some industry players; contagion effects also could greatly affect the industry.

2.10 ASSA, through Advisory Practice Note (APN) 110 and its predecessors has sought to give actuaries clarity on the matter. APN110 replaced its predecessor Professional Guidance Note (PGN) 110. From the first PGN issued in 2003, ASSA has required that the nature and extent of risk inherent in the guarantees be appropriately recognised with a balance being struck between the practicality and complexity of the methodology adopted. The emphasis of the APN remains the calculation of reserves that serve as a buffer to the guarantees should they be realised.

2.11 The APN notes that the guarantees are closely linked to the derivatives traded in the financial markets, the only material difference being that they are embedded in life insurance policies. It recommends "... the use of market consistent stochastic models to quantify reserves required to finance shortfalls in respect of embedded investment derivatives." (ASSA, 2012a: 1) The variables that affect the future liabilities should be simulated stochastically and the future liabilities due to the guarantees projected to the maturity date. The present value of any shortfalls arising therefrom are then to be treated as the reserve, in line with the spirit of the guidance. Though the guidance alludes to a link between the embedded investment

derivatives and the financial markets, it allows discretion for the actuary to ensure that the insurer is adequately protected at all times.

2.12 Hill, Visser & Trachtman (2008), in research commissioned by the Society of Actuaries (SOA), discuss the stochastic pricing of these embedded derivatives. The necessity of such a step is noted by the authors given changing market factors, such as low interest rates that have led to the realisation of some of the guarantees, and competitive pressures that have led to the increasing incorporation of guarantees in life insurance products.

2.13 The authors use the Black–Scholes (BS) model to assist in pricing the GMAB embedded derivative, which has a payoff pattern similar to that of a put option. As the markets become more unpredictable and guarantees more complex, the ability to use the BS method becomes harder and stochastic techniques become more important. Scenarios used must be justifiable. Hill, Visser & Trachtman (op. cit.) discuss three categories of scenarios that could be adopted: historical, long-term risk-neutral and market-consistent. Of the three, the last approach is considered a viable approach for calculating the price and hedging costs of the derivative, given that the value is expected to be linked, directly or indirectly, with other financial instruments trading in the financial markets.

2.14 In the use of stochastic simulation, it is crucial that the number of scenarios used be large enough to allow for a comprehensive and accurate picture on the embedded derivative to be attained. This can be done with relative ease when the underlying distribution is well tabulated with desirable statistical properties but, in other instances, such as long-tailed distributions, “the number of scenarios, even focusing on the average, may need to be much larger.” (Hill, Visser & Trachtman, op. cit.: 27). These authors discuss the use of the so-called Greeks in assessing the sensitivity of the option to certain market factors.

2.15 The adoption of a stochastic approach in the valuation is justified by the unpredictability of most of the factors that affect the option value including interest rates, equity returns, and policyholder behaviour. These need to be given a stochastic consideration in both the pricing and hedging of the option. Hill, Visser and Trachtman (op. cit.) are of the opinion that the expectations of pricing actuaries are changing in this regard. Against the common convention of insurance companies, the spirit of this paper is the use of financial market tools to assist in the pricing and hedging of these options. The use of stochastic reserving in the recent past by the insurance and actuarial community is seen as a step in this direction.

2.16 This step is not in vain since, in financial parlance, insurance contracts can be seen as non-linear instruments with the underlying factor that leads to the exercise of the option, usually mortality, being non-linear in nature. The insurer thus holds what are effectively short option positions on selling a product with guarantees. Reserving has been the past practice in hedging such positions; employing hedging techniques in modern financial engineering to contain risks is a promising recent development.

2.17 Czernicki & Maloof (2008) discuss the topic of variable annuity (VA) guarantee hedging from the premise that volatility in the markets has made it a necessity. They note that some insurers have used first-order Greeks to try to hedge themselves but that this protection is only helpful for small market moves. Concerning the fine line between competitive products and perfectly hedgeable products, most insurers tend to offer competitive but complex products that are a challenge to hedge. The authors note the use of control-variate techniques and fund mapping as new but encouraging developments in meeting this challenge. The importance of adopting hedging as a key consideration during the VA product design phase is emphasised over its use as a post-product design consideration. Such an approach demands a thorough understanding of the guarantees and the corresponding methodologies to price and hedge them.

2.18 Langley et al. (2011) note the upward trend on VA issuance. The financial crisis has, however, meant that hedging costs have increased dramatically. The question of whether hedging is still justifiable in this context is important. There are, however, companies that have displayed the benefits of hedging in the same period—Prudential Insurance, the US life company being a case in point. During the 2008 crisis, the insurer was able to reduce losses from its guarantees through efficient hedging. Langley et al. (op. cit.: 2) note, “... Prudential’s hedging program is one example of how effective hedging can minimise the impact of changes in the value of GMWB guarantees.”

2.19 In the same period, however, other insurers utilising VA have had to retreat or totally exit the market. Erstwhile US industry leaders AXA and ING were victims of market changes. The necessity of pricing and hedging the guarantees appropriately is vital (Langley & Preston, 2012). These authors note the importance of identifying the key factors affecting the guarantee, devising plausible future scenarios and using available derivatives, from simple to complex, to hedge the adverse scenarios.

2.20 Declining interest rates or equity price levels as well as increasing volatility of the equity, forex and treasury markets are identified as factors leading to increasing hedge costs for the guarantees and thus the need to price using a framework that caters for these possibilities. In a paper delivered at the Actuaries’ Club of Hartford/Springfield meeting, Heurtelou (2012) notes examples of capital market products that can be used to hedge against adverse eventualities if they were to occur. Though risks do exist that cannot be hedged using the financial markets, the author notes that most of the risk sources in the VA guarantees can be hedged using available tools such as equity futures, treasury futures and interest rate/equity variance swaps.

2.21 The goal of this paper is to consider some of the propositions above in a South African context.

### 3. EMBEDDED OPTIONS IN GMxB PRODUCTS

The options that lie embedded in the GMxB contracts discussed above have a cost to the company usually referred to as the cost of a guarantee. This can be a maturity, surrender or death guarantee where the cost crystallises on contract expiry, surrender or policyholder death respectively. The fundamental mathematical concepts related to these guarantees are described in this section.

#### 3.1 Maturity Guarantee

3.1.1 The contract in this case guarantees a payoff that is at least:

$$(1 + g_M)^T P; \quad (1)$$

where:

$g_M$  is the per time unit guarantee rate; and  
 $P$  is the initial single premium.

3.1.2 If at maturity the investment by the insurance company,  $I_T$ , is less than this then the cost to the company at maturity time  $T$  is given by:

$$\left\{ (1 + g_M)^T P - I_T \right\}. \quad (2)$$

3.1.3 If the investment value is greater, then the cost to the company is zero since the investment is more than sufficient to pay the guarantee (i.e. the cost to the company is the excess amount the insurer has to pay above the account value if the latter falls short).

3.1.4 The expected value of the cost to the company at maturity denoted  $V_M(T)$  is then given by the expression:

$$V_M(T) = (1 - {}_T s_x) \cdot (1 - {}_T q_x) \cdot \text{MAX} \left\{ 0, (1 + g_M)^T \cdot P - I_T \right\}; \quad (3)$$

where:

${}_T q_x$  is the probability of a life aged  $x$  dying within the next  $T$  years; and  
 ${}_T s_x$  is the probability of a life aged  $x$  surrendering within the next  $T$  years.

#### 3.2 Surrender Guarantee

3.2.1 The surrender guarantee undertakes to pay the policyholder a proportion of the original premium  $P$ :

$$(1 + g_S)P, \quad (4)$$

with  $g_S < 0$  to dissuade policyholders from surrendering too soon. The surrender rate is frequently modelled as a function of duration in force and policy term. For simplicity, the representation used in this section shows the  $g_S$  at the time of surrender with the policy duration and term having already been implicitly considered in arriving at this rate and hence suppressed in the notation.

3.2.2 If at any time the policy is surrendered and the value of the investment,  $I_t$ , is less than the surrendered amount then the extra payout by the company is given by:

$$\{(1 + g_s)P - I_t\}. \tag{5}$$

3.2.3 The fair charge by the insurer today for a surrender that occurs at the end of year  $t \in \mathbb{N}$  denoted  $V_S(0)$  is then given by:

$$V_S(0) = {}_1s_{x+t-1} \cdot (1 - {}_tq_x) \cdot \text{MAX}\{0, (1 + g_s)P - I_t\} e^{-\int_0^t r_u du}; \tag{6}$$

where:

$r_u$  is the risk-free rate of interest.

### 3.3 Death Guarantee

3.3.1 The death guarantee amount that is paid to the estate for death that occurs at a random time  $t$  is at least:

$$(1 + g_D)^t P; \tag{7}$$

where:

$g_D$  is the per time unit guarantee rate; and  
 $P$  is the initial single premium.

3.3.2 If death occurs at such a time  $t$  when the value of the investment,  $I_t$ , is less than this amount then the company has to make an extra payout of:

$$\{(1 + g_D)^t P - I_t\}. \tag{8}$$

3.3.3 The fair charge by the insurer today for the guarantee, on a death that occurs at the end of year  $t \in \mathbb{N}$ , denoted  $V_D(0)$  is then given by:

$$V_D(0) = (1 - {}_{t-1}s_x) \cdot (1 - {}_{t-1}q_x) q_{x+t-1} \cdot \text{MAX}\{0, (1 + g_D)^t P - I_t\} e^{-\int_0^t r_u du}. \tag{9}$$

3.4 The maturity or death amounts, denoted  $X$ , can be expressed in a continuous compounding sense as:

$$X = e^{gT} P; \tag{10}$$

where:

$g$  is the per time unit guarantee rate for the respective guarantee.

3.5 The guarantees have payoff profiles very similar to vanilla and exotic options that are widely traded in the financial markets; the main difference is that the traded options have significantly lower durations. This distinction poses a challenge because of the lack of long-dated traded instruments with which to calibrate when pricing. This notwithstanding, the payoff structure enables us to formulate a well-known expression from which insightful results can be derived. In particular, the GMxBs are put options whose maturity date is known for the maturity guarantee but random for the surrender and death guarantees. The payoff at such a maturity time,  $T$ , for the holder of the option with strike price  $X$  and underlying  $S_T$  is expressed as:

$$(X - S_T)_+ = \text{MAX}\{0, X - S_T\}. \quad (11)$$

3.6 If we consider the payout to be made at maturity time,  $T$ , and a guarantee rate of  $g$  per unit time for the GMxB in question, the policyholder receives:

$$Y_T = \text{MAX}(e^{gT}P, I_T) = e^{gT} \text{MAX}(P - e^{-gT}I_T, 0) + I_T. \quad (12)$$

3.7 The value of the guarantee at  $T$  is the account value  $I_T$  added to a put option with an exercise of  $P$  and the underlying being the discounted account value.

3.8 Further, the guarantees can be a return of premium, roll-up or the rising-floor guarantee. In this report, it is assumed that the guaranteed benefit does not change as the fund value changes or, if it does, it is a compound growth through a constant interest rate, essentially an interest guarantee. The policyholder also normally has the choice regarding which product to choose and in some instances the investment strategy to be followed. The insurance company should therefore be able to protect itself from adverse exercise of any of the above products.

#### 4. PRICING EMBEDDED GUARANTEES IN A VARIANCE-GAMMA ECONOMY

4.1 The VG process is a Lévy process based on the notion of random time which was derived in Madan & Seneta (1990) and discussed more generally in Madan, Carr & Chang (1998). In the former case, the authors discuss the symmetric VG process whereas, in the latter, the general non-symmetric VG process is considered. The two processes meet the fundamental expectation of any market model, that it be arbitrage free. The log price is expressed as a Brownian motion considered at a random time. Furthermore, they incorporate long-tailedness and the general case takes skewness into account.

4.2 Joshi (op. cit.) discusses the VG process and notes some useful properties of this process, key among them its attribute as an accurate model for stock price movements in the small scale. The author notes that the movement of stock prices on small time scales is not really similar to a Brownian motion. Instead, these prices move in little jumps rather than

continuously with the total number of up and down moves being finite rather than infinite. The VG model is able to capture this together with its natural adaptation in option pricing to both Monte Carlo and replication pricing techniques.

4.3 The VG process has been given a number of representations. Two such representations from Schoutens (2003), useful for the context under consideration, are discussed in sections 4.3.1 and 4.3.3 below:

4.3.1 DEFINITION 1

The VG process  $X^{(VG)} = \{X_t^{(VG)}, t \geq 0\}$  is the process such that:

- $X_0^{(VG)} = 0$ ,
- The process has independent and stationary increments,
- The increment  $X_{s+t}^{(VG)} - X_s^{(VG)} \sim VG\left(\sigma\sqrt{t}, \frac{\nu}{t}, t\theta\right)$  i.e. the increment follows a VG distribution.

4.3.2 DEFINITION 2

A gamma process is a Lévy process with independent increments drawn from a gamma distribution. A stochastic gamma process,  $G(t)$ , is defined with the following properties:

- $G(0) = 0$ ,
- $G(t+s) - G(t) \sim \Gamma(\gamma s, \lambda)$  for any  $s, t \geq 0$ ,
- $G(t)$  has independent increments  $G(t_n) - G(t_{n-1})$  where  $t_n > t_{n-1}$

4.3.3 DEFINITION 3

Let  $G = \{G_t, t \geq 0\}$  be a gamma process with parameters  $a = \frac{t}{\nu}$ ,  $\nu > 0$  and  $b = \frac{1}{\nu}$ ,  $\nu > 0$

and let  $W = \{W_t, t \geq 0\}$  be a standard Brownian motion. Further, let  $\sigma > 0$  and  $\theta \in \mathbb{R}$ , then the VG process is defined as:

$$X_t^{(VG)} = \theta G_t + \sigma W_{G_t}. \quad (13)$$

4.4 In this case, the parameters  $\theta$  and  $\nu$  are useful in controlling for skewness and kurtosis respectively with  $\theta$  accounting for skewness in the distribution and  $\nu$  accounting for the excess kurtosis relative to the normal distribution. It is worth noting that with a different parameterisation, Madan, Carr & Chang (op. cit.) show the VG process as a difference of two independent gamma processes. The interested reader is referred to their seminal paper. This parameterisation is quite useful in simulation contexts. (Refer to the appendix where we provide a discussion on simulation.)

## 4.5 Model Theory

4.5.1 Madan, Carr & Chang (op. cit.) derive the risk-neutral underlying price dynamics under a VG economy as:

$$\begin{aligned} X_t^{(VG)} &= \theta_{VG} G_t + \sigma_{VG} W_{G_t}; \\ S(t) &= S(0) e^{(r-q+\omega)t + X_t^{(VG)}}; \end{aligned} \quad (14)$$

where:

$$\omega = \frac{1}{\nu} \ln \left( 1 - \theta_{VG} \nu - \frac{\sigma_{VG}^2 \nu}{2} \right),$$

$\theta_{VG}$  is the drift of the diffusion part,

$\sigma_{VG}$  is the volatility of the diffusion part,

$\nu$  is the variance rate of the gamma part.

4.5.2 It is worthwhile to comment on the change of measure from  $\mathbb{P}$  to  $\mathbb{Q}$  in deriving the risk-neutral price. The Mean Martingale Correcting Term (MMCT) is the approach used in the change of measure. The MMCT has been shown to be a special case of the Esscher transform approach (see Miyahara, 2004). Using the characteristic function approach, he shows that to obtain the MMCT change of measure, we shift the VG process  $X_t$  to  $X_t + \omega t$  and in terms of the characteristic function, we have:

$$\phi_{X_t}^{\mathbb{Q}}(u) = \phi_{X_t}(u) e^{\omega t}. \quad (15)$$

4.5.3 If we assume that the dividend yield  $q$  is incorporated in the stock price, which is a reasonable assumption in an option pricing context, we have:

$$S(t) = S(0) e^{(r+\omega)t + X_t^{(VG)}}. \quad (16)$$

4.5.4 Denoting the risk neutral measure with  $\mathbb{Q}$ , the value of a European call option is then given by:

$$c_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[ (S(T) - K)^+ \right]. \quad (17)$$

4.5.5 After taking the expectation, Madan, Carr & Chang (op. cit.) arrive at the expression for the time 0 price of the option as:

$$\begin{aligned} c(S(0); K, T) &= S(0) \Psi \left( d \sqrt{\frac{1-c_1}{\nu}}, (\alpha + s) \sqrt{\frac{\nu}{1-c_1}}, \frac{T}{\nu} \right) \\ &\quad - K e^{-rT} \Psi \left( d \sqrt{\frac{1-c_2}{\nu}}, \alpha s \sqrt{\frac{\nu}{1-c_2}}, \frac{T}{\nu} \right); \end{aligned} \quad (18)$$

where:

$$d = \frac{1}{s} \left[ \ln \left( \frac{S(0)}{K} \right) + rT + \frac{T}{\nu} \ln \left( \frac{1-c_1}{1-c_2} \right) \right],$$

$$\zeta = -\frac{\theta}{\sigma^2}, \quad s = \frac{\sigma}{\sqrt{1 + \left(\frac{\theta}{\sigma}\right)^2 \frac{\nu}{2}}}, \quad \alpha = \zeta s,$$

$$c_1 = \frac{\nu(\alpha + s)^2}{2}, \quad c_2 = \frac{\nu\alpha^2}{2},$$

and  $\Psi$  represents the modified Bessel function of the second kind and a degenerate hypergeometric function.

4.5.6 The put-call parity relation is applied to the call option price in Hirsra & Madan (2001) and the resultant price of a European put option is:

$$p(S(0); K, T) = Ke^{-rT} \Psi \left( -d \sqrt{\frac{1-c_2}{\nu}}, \alpha s \sqrt{\frac{\nu}{1-c_2}}, \frac{T}{\nu} \right) - S(0) \Psi \left( -d \sqrt{\frac{1-c_1}{\nu}}, (\alpha + s) \sqrt{\frac{\nu}{1-c_1}}, \frac{T}{\nu} \right). \tag{19}$$

4.5.7 The application of the VG process in the pricing of options has been carried out by a number of authors in the context of American and European options; see, for example, Madan, Carr & Chang (op. cit.), Moosbrucker (2006) and Wang (2009). In the discussion that follows we consider the VG process in the pricing of embedded derivatives in life insurance products, in particular the GMxB products.

4.6 The methodology used in calibrating the VG model is discussed in the appendix to this paper (see Section A.3). The results from this methodology are shown in Table 2. The Maximum Likelihood Estimation (MLE) approach is applied to South African monthly returns data from and estimates of the three VG model parameters derived from this.

TABLE 2. Final MLE VG parameter estimates

Parameter	$\theta$	$\nu$	$\sigma$	$\mu$
Estimate	-0,0148	0,4461	0,0544	0,0234

#### 4.7 Model Application to European Type Options

4.7.1 Kling, Ruez & Rub (2010) discuss product valuation under the assumption of independence between the financial markets and demographic mortality. This is a reasonable assumption that can be checked through a correlation test. If the insurer is risk neutral with

respect to mortality risk then we are able to use the financial market risk-neutral measure and the mortality measure in the GMxB valuation. Finally, the reasonable assumption that the underlying account is linked directly to a market index, as discussed in Coleman et al. (2005) is made, in this case the ALSI, and used to price the costs in the South African context. A discussion of how to obtain the option price from a simulation is given in the appendix.

4.7.2 The assumption of risk-neutral mortality results in a European-type option which can be priced with the VG option price formula. This is based on the extra cost to the insurer above the investment value  $\max(f(g, P) - I_T)$ . In this expression,  $f(g, P)$  is the guaranteed amount which, if we ignore mortality, is a function of the guarantee rate and the initial premium, used to calculate the charge for the embedded option.

#### 4.7.3 APPLYING THE MODEL TO PRICING

4.7.3.1 The application of the VG model to European-type options follows. The formula for the option price is based on the nature of the payoff and the assumption that, under the risk-neutral measure, the one-period account value log-returns are independent random variables that follow a VG distribution:

$$\ln\left(\frac{I_t}{I_{t-1}}\right) = r_t \sim VG(\sigma, \nu, \theta).$$

4.7.3.2 This is applied for a European put option with a strike amount of R1000 using three different methods: analytical, integration and Monte Carlo simulation, set out in Table 3.

4.7.3.3 The analytical pricing approach is as discussed in Madan, Carr & Chang (op. cit.). The integration method involves brute force integration as discussed in Rebonato (2004). These two approaches require a numerical integration method and are quite sensitive to the method used. In the case at hand, though, the integral is an infinite integral. The numerical integral evaluation is done using *quadgk*, the Gauss–Kronrod quadrature in Matlab which is noted as an efficient and accurate approach, especially for infinite intervals and cases where integrands contain singularities at endpoints.

4.7.3.4 Binkowski (2008) discusses the errors introduced by such an integration cut-off and the choice of the method used. Binkowski notes that the errors are considered minimal with the prices obtained being accurate to at least three decimal places. This is considered sufficient accuracy for this research.

4.7.3.5 The analytical and brute force integration method prices product comparable results, as shown in Table 3. The MC method requires a significant number of simulations and computer time for the same answer for short-tenor options under the VG framework and hence would not be the preferred method in such cases. However, for large tenors and due to the failure of numerical integration methods to converge, the first two methods fail. Since embedded options tend to have long maturities, the Monte Carlo approach is used hereafter for their pricing.

4.7.3.6 In Table 3, a simulation of 10000 stock price possibilities at maturity was carried out for 100 runs and the average taken. A Monte Carlo simulation of 10000

possible stock index values after 10 years starting from R1000 is shown in Section A.2 of the appendix.

TABLE 3. European put option prices under a VG economy

1-year European put option			5-year European put option			10-year European put option		
$S_0$	Method	Estimated option price	$S_0$	Method	Estimated option price	$S_0$	Method	Estimated option price
R500	Analytical	399.8171	R500	Analytical	143.2720	R500	Monte Carlo	41.4081
R750		163.3511	R750		50.4111	R750		14.7013
R1 000		33.1087	R1 000		17.2321	R1 000		5.7913
R1 250		4.1009	R1 250		6.0712	R1 250		2.5341
R1 500		0.4288	R1 500		2.2467	R1 500		1.1630
R500	Monte Carlo	399.8015	R500	Monte Carlo	143.2635			
R750		163.1880	R750		50.3417			
R1 000		33.1958	R1 000		17.2493			
R1 250		4.1272	R1 250		6.0300			
R1 500		0.4358	R1 500		2.2177			
R500	Integration	399.8171	R500	Integration	143.2721			
R750		163.3511	R750		50.4112			
R1 000		33.1087	R1 000		17.2323			
R1 250		4.1009	R1 250		6.0715			
R1 500		0.4288	R1 500		2.2470			

4.7.3.7 The output shows the ability of the MC approach to simulate stock price ranges across a wide spectrum in a VG economy. Though most of the simulated values lie within reasonable ranges expected in normal market times, the ability to simulate extremes (so-called heavy tailedness) and thereby capture the possibility of the markets performing exceptionally well or very poorly during periods of turbulence accommodates a desired quality of any market model.

4.7.4 A discussion of the parameter sensitivity analysis is given in Section A.3.11 of the appendix. The analysis shows the three VG parameters to be insensitive to modest perturbations on the base rates. In situations where this is not the case, the option prices obtained still lie within the 95% confidence interval. A summary of this result is contained in Table 4 (overleaf) based on a 10-year European put option (with both  $S_0$  and strike price R1000) where one of the parameters is adjusted and the others are held constant.

#### 4.8 Variance-Gamma Model applied to pricing the GMxBs

4.8.1 The mortality rates from ASSA given in the Section A.6 of the appendix are applied in this section to evaluate the GMxBs prices.

TABLE 4. VG option price sensitivity as the parameters vary

	10-year European put option		
	$\theta$	$\nu$	$\sigma$
Base scenario + 5%	7,7436	5,8851	59,0012
Base scenario + 1%	5,2948	5,8166	12,4292
Base scenario	5,7913	5,7913	5,7913
Base scenario – 1%	6,7502	5,7722	1,8476
Base scenario – 5%	16,3129	5,7148	0

#### 4.8.2 GUARANTEED MINIMUM MATURITY BENEFIT

4.8.2.1 We assume that the guarantee is a roll-up maturity guarantee with a rate of  $g_M$  per year and that the policyholder neither surrenders nor withdraws from the account.

TABLE 5. 10-year 5% GMMB charges under the VG economy

Age ( $x$ )	$g_M = 5\%$	
	Male	Female
50	24,3212	26,3395
55	20,9931	22,5787
60	18,9030	22,5076
65	16,8520	22,7264

TABLE 6. 10-year 10% GMMB charges under the VG economy

Age ( $x$ )	$g_M = 10\%$	
	Male	Female
50	105,6165	114,3808
55	91,1639	98,0494
60	82,0873	97,7408
65	73,1810	98,6907

4.8.2.2 The figures below show price behaviour on a stand-alone and a comparative setting using the regime switching results discussed in the technical report by Ngugi, Maré & Kufakunesu (op. cit.). The regime switching model has been shown by Hardy (2001) to be an excellent fit for North American equity data. The behaviour of the equity markets is heavily influenced by the macroeconomy and hence a natural explanation of the appeal of the model in modelling equity returns besides its tractability. As Hardy (op. cit.: 173) notes of the model “... it does as good a job as can be done.”

4.8.2.3 From Figure 1 it is apparent that a higher guarantee rate results in a higher guarantee charge and, owing to higher mortality rates for males relative to females, the charges for the latter are higher given the same starting age  $x$  across all the policy inception

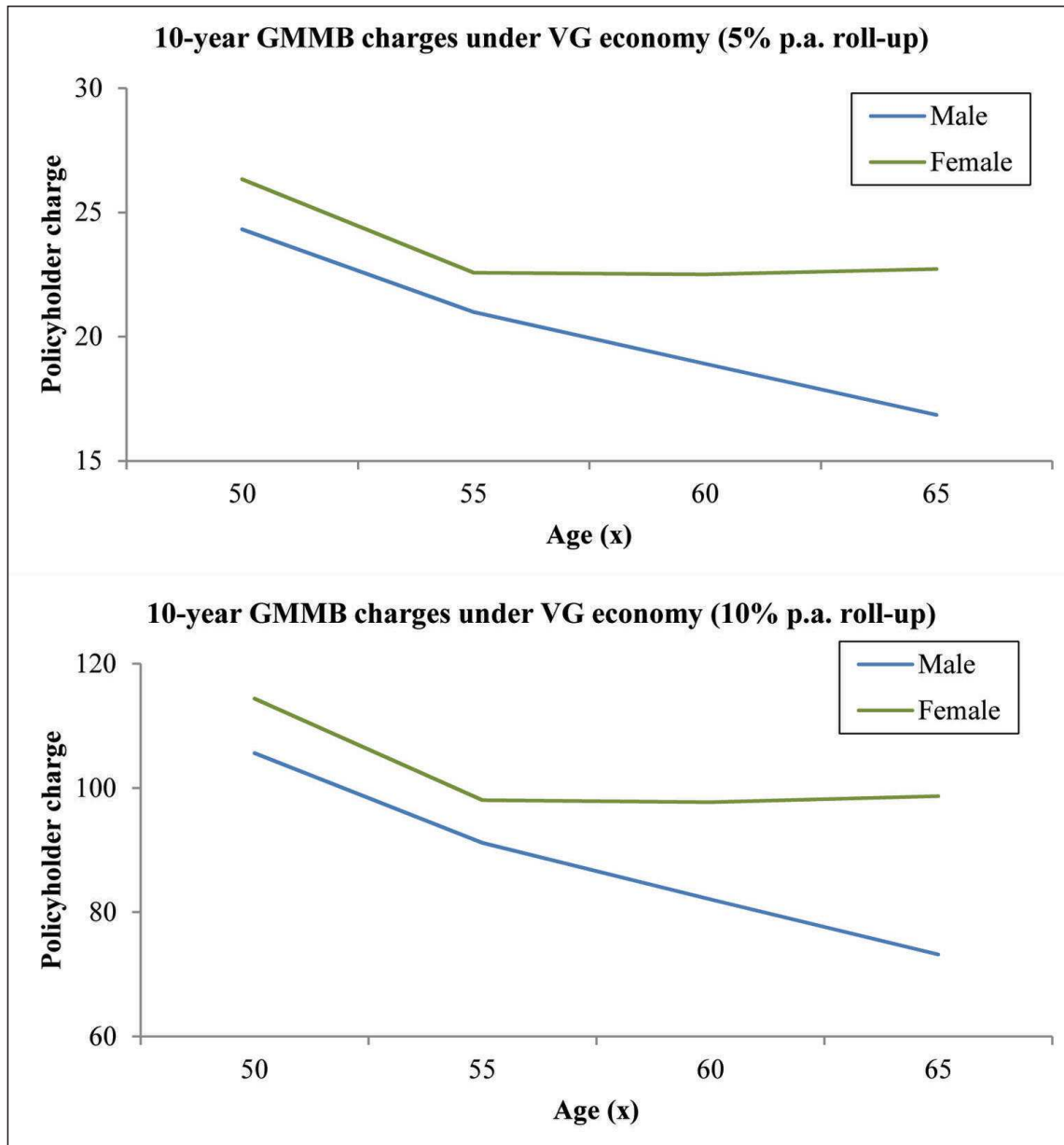


FIGURE 1. 10-year GMMB charges under the VG economy

ages considered. A comparative plot is shown in Figure 2 from which it is noteworthy that when the guarantee is 5% the charges lie between the regime 1 and regime 2 charges obtained in the RS model.

4.8.2.4 If the guarantee rate is 10%, however, the charges obtained under the VG economy are slightly higher than the corresponding charges obtained in the RS setting as shown in the plots below. This is explained by the fact that a higher guarantee rate increases the probability that the index value at maturity proves to be less than the guaranteed amount at the time. The chances of the option ‘exercise’ are thus higher and this should result in a higher charge. In the RS model, approximately 60% of the time, the economy is in regime 1 (low

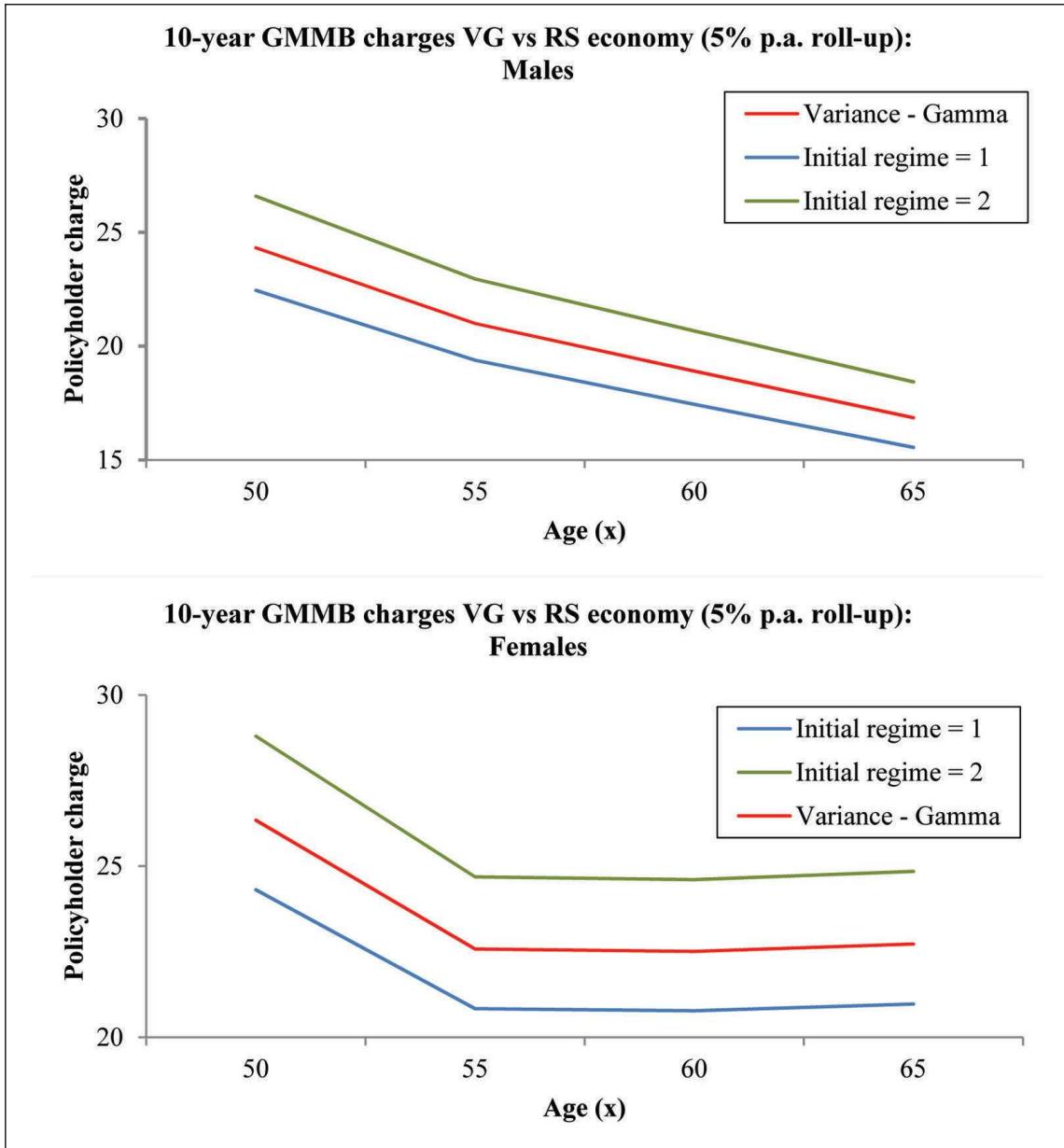


FIGURE 2. 10-year 5% GMMB charges: VG versus RS economy

volatility, high return regime) and this could also have filtered through the index dynamics resulting in the RS model having higher values for  $S_T$  than the VG model. The VG model assumes the existence of one economy but captures extremes at the tails.

#### 4.8.3 GUARANTEED MINIMUM DEATH BENEFIT

4.8.3.1 The GMMB charges calculated using the ASSA basis are as shown in Table 7.

4.8.3.2 In the case where the guarantee rate is 0%, the charge is highest if death occurs in the first year, falling thereafter. This is true for both males and females. The charges for the former are higher compared to their female counterparts for the same age at inception

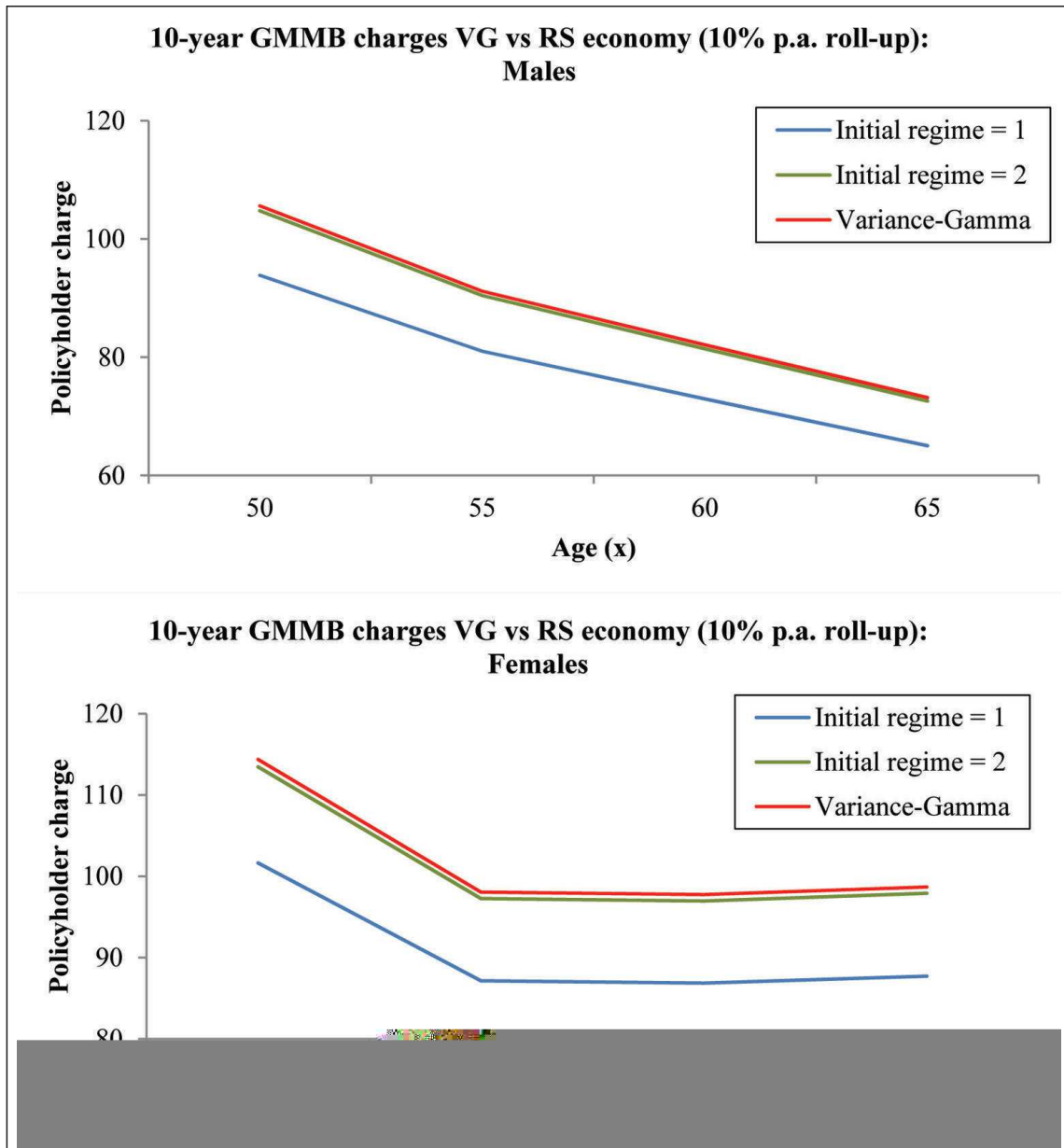


FIGURE 3. 10-year 10% GMMB charges: VG versus RS economy

of a policy. This is attributable to the higher mortality rates of males relative to females and hence a higher likelihood of the guarantee maturing.

4.8.3.3 If the insurer is in a VG economy with an annual guarantee rate of 5%, the guarantee charge is highest if death occurs in the 10th year. This illustrates the impact of the guarantee. Death in any period before this would be less expensive to the company.

4.8.3.4 A comparison of the VG GMMB maximum charges with those obtained in a regime-switching economy shows the VG charges sandwiched between regime 1 and regime 2 charges. This is illustrated in Figure 4.

TABLE 7. 10-year 0% GMDB charges under the VG economy

Death (maturity) in:	$g_D=0\%$			
	Female		Male	
	Age (x) = 50	Age (x) = 60	Age (x) = 50	Age (x) = 60
1st year	0,093 730	0,229 853	0,207 994	0,508 723
2nd year	0,093 597	0,229 56	0,207 073	0,503 681
3rd year	0,086 108	0,210 885	0,189 687	0,456 727
4th year	0,077 493	0,189 448	0,169 795	0,402 162
5th year	0,068 54	0,167 597	0,149 609	0,346 999
6th year	0,060 654	0,147 854	0,131 569	0,297 575
7th year	0,052 923	0,128 733	0,1142	0,250 806
8th year	0,046 384	0,112 336	0,099 525	0,211 52
9th year	0,040 585	0,097 637	0,086 467	0,177 293
10th year	0,035 261	0,083 933	0,074 551	0,147 081
Maximum charge	0,093 73	0,229 853	0,207 994	0,508 723

TABLE 8. 10-year 5% GMDB charges under the VG economy

Death (maturity) in:	$g_D=5\%$			
	Female		Male	
	Age (x) = 50	Age (x) = 60	Age (x) = 50	Age (x) = 60
1st year	0,093 521	0,229 341	0,207 53	0,507 59
2nd year	0,129 407	0,317 387	0,286 297	0,696 383
3rd year	0,150 95	0,369 688	0,332 527	0,800 657
4th year	0,166 954	0,408 155	0,365 813	0,866 434
5th year	0,178 552	0,436 601	0,389 741	0,903 957
6th year	0,188 185	0,458 73	0,408 205	0,923 253
7th year	0,195 456	0,475 438	0,421 765	0,926 278
8th year	0,201 666	0,488 409	0,432 712	0,919 635
9th year	0,207 301	0,498 71	0,441 658	0,905 578
10th year	0,211 82	0,504 207	0,447 845	0,883 55
Maximum charge	0,211 82	0,504 207	0,447 845	0,926 278

4.8.3.5 There has been a great deal of research on the use of more refined models in the pricing of options. These have been shown to provide better and more realistic price processes for the underlying (see, for example, Hull & White, 1987, Heston, 1993, and Ulmer, 2010). To the extent that such a framework does not deviate from regulatory, accounting and taxation guidelines, it provides a plausible approach to adopt in the pricing of the options. This would be another stride in walking across what Embrechts (op. cit.) refers to as the financial bridge to actuarial pricing. He notes that the correct pricing will be one that incorporates both financial theory and actuarial practice, a theme that this section has sought to present.

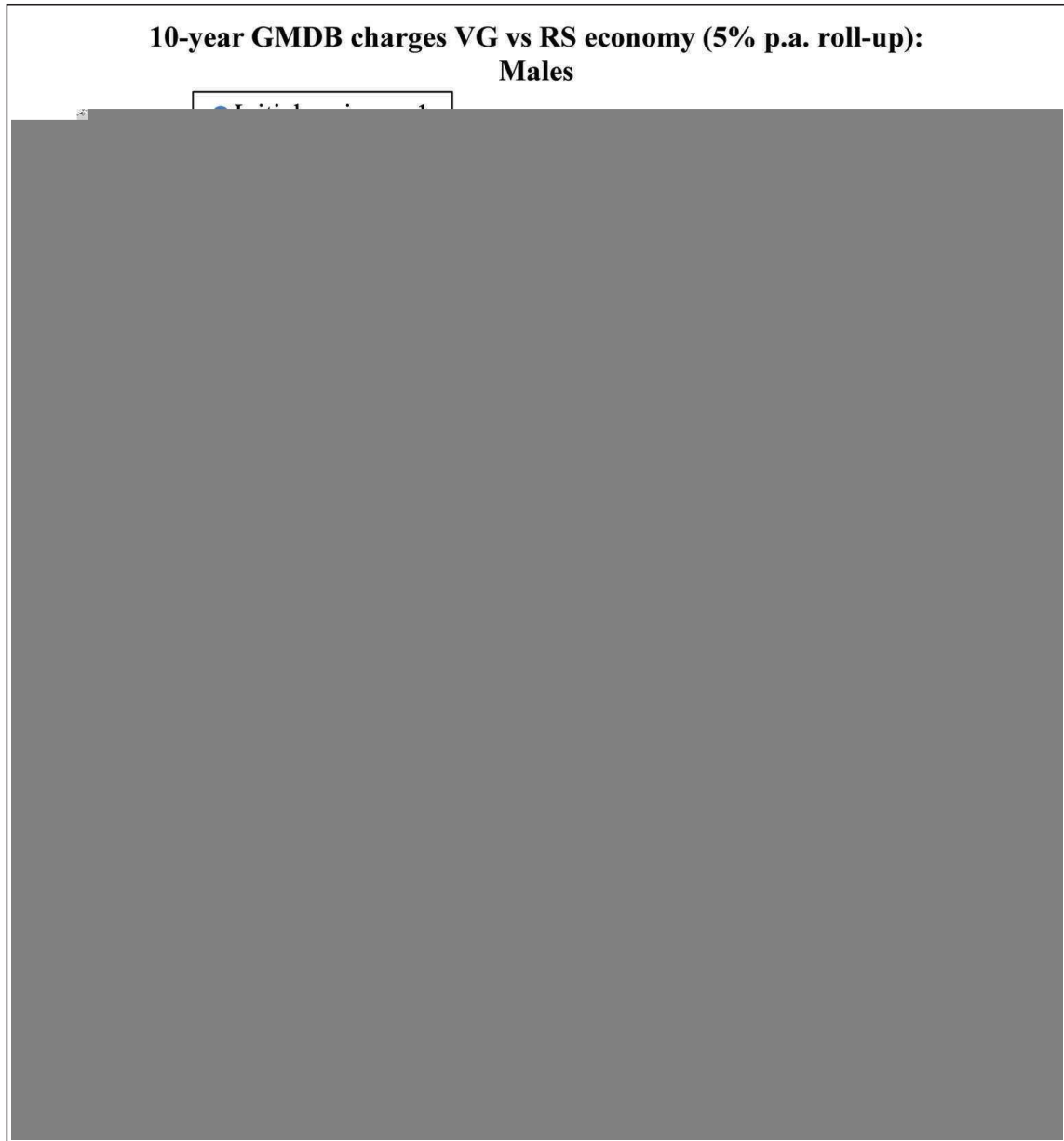


FIGURE 4. 10-year 5% GMDB charges: VG versus RS economy

## 5. EMBEDDED OPTIONS AND CAPITAL MANAGEMENT

5.1 We turn now to the important topic of the capital management of an enterprise with embedded guarantees in its products. De Weert (2011) notes the importance of:

- accounting perspectives, which include the IFRS guidelines, and
- regulatory perspectives which would include the Solvency Assessment and Management (SAM), Solvency II and ASSA APN110 and Standard of Actuarial Practice (SAP) 104.

We consider these two perspectives in the discussion that follows.

## 5.2 Accounting Perspective

5.2.1 The balance sheet of a life insurer is characterised by significant activity on the liability side. This is because the insurer is contingently indebted to the policyholders.

5.2.2 The role of capital management is heavily influenced and guided by the desire to ensure that the insurer holds the required capital in line with the risks taken. This desire, however, is bent by the need to conform to the financial reporting standards, in particular, the International Financial Reporting Standard (IFRS) 4 on Insurance Contracts and IFRS 9 on Financial Instruments.

5.2.3 Both of these standards reflect a move towards fair value accounting, in turn signalling the need for a valuation approach that comprehensively uses asset and liability values that are as close as possible to the market values. IFRS 4 in particular allows the insurer to use a different valuation methodology if such a move results in a move towards market-consistent valuation. This is allowed at a group level, business unit level, and even at a portfolio level. Given the goal of increasing the reliability and prudence of the financial statements, the move towards valuation methodologies that capture the market behaviour appears worthwhile, provided the requisite comprehensive disclosures as required by the International Accounting Standards Board (IASB) are met.

## 5.3 Regulatory Perspective

5.3.1 The life insurance embedded derivatives are governed, from a regulatory perspective, by the Financial Services Board (FSB) SAM guidelines which are comparable to the European Union equivalent of Solvency II and ASSA requirements.

### 5.3.2 ACTUARIAL SOCIETY OF SOUTH AFRICA GUIDELINES

5.3.2.1 SAP 104, issued by ASSA, deals with the calculation of the value of assets, liabilities and capital adequacy requirement of long-term insurers (ASSA, 2012b). It is obligatory for long-term insurance statutory actuaries.

5.3.2.2 The Standard distinguishes between three forms of reporting which may dictate different valuation methodologies: financial reporting, statutory reporting and tax reporting. Although no particular method is imposed, the principles of the financial soundness valuation are deemed a key guide along with the need to refer to the complementary APN110.

5.3.2.3 Advisory Practice Note (APN) 110, also issued by ASSA, deals with the allowance for embedded investment derivatives. It notes that a deterministic approach to valuing the embedded derivatives falls short of real-world random behaviour and calls for a stochastic method in the valuation.

5.3.2.4 APN110 does not prescribe any particular stochastic model to aid in the valuation of the embedded derivatives. The overriding theme, however, is that the model adopted by the actuary must be market-consistent and sufficiently robust for reliable quantification of reserves that will meet the costs of the guarantees when they are realised. The guidance notes that, due to the long-term nature of life insurance, calibration using tradable instruments is a challenge but, that, whenever possible, the actuary should use historical analysis complemented with implied parameters for tenors where tradable derivatives exist.

The guidance recommends back testing on the adopted model to ensure that it reproduces the observed traded derivatives. This adds to the robustness and credibility of the model created. If the Monte Carlo approach is used, then a minimum of 2000 simulations is recommended.

5.3.2.5 The need for precision, though a necessary consideration, must not be achieved at the expense of practicality. The guidance note stresses that recognising the nature and extent of risk in an embedded derivative is more important than mathematical precision if the two are in conflict:

Some guarantees offered by life offices could be very complex instruments. As such, they may be very difficult to model precisely. Parameter estimation may often also be problematic. The actuary needs to bear in mind that the appropriate recognition of the nature and extent of risk involved in those guarantees is more important than surgical precision in the valuation models. For this reason, the actuary must use his/her judgement to strike an appropriate balance between complexity and practicality. (ASSA, 2012a:8)

5.3.2.6 Finally, the APN notes that it is important to test the approach used to shock scenarios that can adversely affect the insurer.

### 5.3.3 FINANCIAL SERVICES BOARD AND SOLVENCY II GUIDELINES

5.3.3.1 The FSB in South Africa is tasked with the role of instituting regulatory capital requirements for, inter alia, life insurers. Embedded derivatives fall within this scope and currently the SAM directive provides a guide. The European Union equivalent, Solvency II directive, provides a good comparative base.

5.3.3.2 The directives are based on three pillars. Pillar 1 deals with the quantitative requirements, Pillar 2 the governance and risk management frameworks, and Pillar 3 with disclosure and transparency, essentially market discipline. The valuation of embedded derivatives falls under Pillar 1, given that this valuation is mainly a quantitative exercise.

5.3.3.3 Article 79 under the Solvency II system requires that the value of any embedded guarantees be taken into account in the valuation and that "... the assumptions used shall take account, either explicitly or implicitly, of the impact that future changes in financial and non-financial conditions may have on the exercise of those options." (CEIOPS, 2009: 25). The SAM directives have the same spirit.

5.3.3.4 SAM has a bias towards the use, in valuation, of risk-neutral frameworks though the directive does note that this may not be adequate for investment guarantees. Deviations from the risk-neutral setting are allowed, provided such a move does not result in market-inconsistent values. Finally, the directive requires that all decrements and risk-drivers that materially affect the guarantees be considered in the valuation. This research focuses primarily on pricing. Consideration of the implications of decrements such as lapses and surrender falls outside of the scope of this research.

5.4 The consideration of correlation from a risk management perspective is discussed in Joubert & Langdell (2013), who note that the correlation matrix can provide a useful starting point in an endeavour to assess decrements and risk drivers. The authors provide a

useful discussion on how to correct constructed matrices that mathematically fall short of the requirements for a correlation matrix. The use of such an approach in the holistic assessment of life insurance capital management issues is likely to address any demerits of a stand-alone consideration of risk drivers in any setting.

5.5 We note the consistency of the capital management practices of three major South African life insurance companies with these principles, as disclosed in their annual reports. The insurers largely follow the guidelines noted and discussed earlier with considerable use of sensitivity analysis in their valuations to judge the exposure to economic and other unfavourable shocks.

5.6 This section has considered the qualitative aspects needed in the valuation of a life insurer's liabilities as set out in various guidelines. The goal of this assessment was to gain an understanding of how these guidelines fit into the considered frameworks in the research. Overall, the frameworks considered support the need for prudent, sound and market consistent models needed in valuation.

## 6. CLOSING REMARKS

6.1 It is a well-accepted premise in quantitative finance that asset prices display many small jumps. This suggests that the quantitative model chosen to assess the characteristics of such a series should be aligned with this understanding. Published research on the choice of models is, however, inconclusive. This is what has guided the discourse of this research, in particular, to answer whether:

- the guarantees embedded in variable annuities, more specifically GMxBs, can be effectively priced using a suitably chosen exponential jump model, and whether
- this approach might also meet the expectations of regulators and other stakeholders on the life insurer's approach to risk and capital management.

6.2 The Variance-Gamma (VG) model is a natural choice for the jump model given its ability to generate an infinite number of jumps within any finite interval and, together with its related offshoots, it has provided the frameworks to address these questions.

6.3 In considering the results from Section 4, theoretically-improved models can provide a pricing framework and conclusions not significantly different from current models that assume a normal distribution for the underlying returns. The theoretically-improved models have the added advantage of incorporating stylised facts on financial markets behaviour and thus give a truer picture over time. The use of this approach would thus provide benefits to the insurer from more of an asset-liability perspective.

6.4 The variable annuity industry is noted by many authors as one of the most complex in the life insurance industry because of the riders offered and the link to the financial markets

of these riders. There is, however, little dispute to the proposition that the pricing of options should take into account empirically observed features such as skewness and kurtosis.

6.5 This research has encouraged thinking in this regard. It does not suggest that the frameworks so applied are perfect models, but rather that they are more in tune with reality and the observed dynamics of the South African financial markets.

6.6 The scale of complexity and extent of debate on the subject matter indicates the need for further studies on the topic to ensure that variable annuities are appropriately priced and hedged. This research can be built on by exploring a number of further issues.

6.7 In a forthcoming paper, the authors consider the VG stochastic volatility model in light of the major difficulties of the VG model (and other Lévy processes) which are the assumption of independent increments and failure to incorporate stochastic volatility. Further, they consider the hedging and risk measurement of the embedded options, which are key issues in risk management. The delta-hedging approach is used for the dynamic replication of the derivatives and the Value-at-Risk and Expected Shortfall techniques implemented for the corresponding risk measurement.

6.8 The use of a deterministic interest rate could be changed to assess the impact of stochastic interest rates on the prices. Kijima & Wong (2007), Peng, Leung & Kwok (2009) and Tiong (2013) have considered research in this direction under different frameworks, using a number of the well-known interest rate models. This approach is helpful in explicitly taking into account interest-rate risk in the pricing and would be a further refinement to the framework.

6.9 The study has also made assumptions on the impact of some demographic risks such as mortality and policyholder behaviour. The mortality rates are based on the ASSA tables and though they give a representative picture, the use of a stochastic mortality model may be a viewpoint worth taking. Milevsky & Posner (2001) considered the use of exponential mortality models which could be incorporated into the case at hand or the application of the stochastic mortality models discussed in Cairns et al. (2008).

6.10 The use of quantitative techniques in asset-liability management will continue to be of relevance to the insurance industry for many years to come.

6.11 The early adopters of any new model are normally able to gain competitive advantage and the VG together with its extended frameworks provide such a basis. The costs in this setting will not necessarily be too high for the insurer. The principal conclusion of this research is that the frameworks are tenable.

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**APPENDIX: SUPPORTING INFORMATION**

In this appendix, we discuss the simulation of a VG process and option prices under this process. The calibration of the model using South African data is also presented in this appendix together with a sensitivity analysis on the parameters so obtained. Furthermore, we present the modified Bessel function of the second kind and the degenerate hypergeometric function mentioned in ¶4.5.5.

**A.1 Simulating Variance-Gamma Process**

Monte Carlo simulation is used in the simulation of the embedded guarantees in the GMxB products considered and thus the need to discuss a technique for simulating a VG process. In Definition 3, it was noted that a VG process can be obtained by time-changing a standard Brownian motion with drift by a gamma process. If one can sample these two processes then a sample path for a VG process can then be obtained.

**A.1.1 SIMULATING A GAMMA PROCESS**

Schoutens (op. cit.) discusses the simulation of a gamma process noting that if a random variable  $X \sim Gamma(a,b)$ , then, for  $c > 0$ ,  $\frac{X}{c} \sim Gamma(a,bc)$ . All that is needed is a good generator of a  $Gamma(a,1)$  for which the Berman’s gamma generator is used as follows:

- (1) Generate two independent uniform random numbers  $u_1$  and  $u_2$ .
- (2) Set  $x = u_1^a$  and  $y = u_2^{1-a}$ .
- (3) If  $x + y \leq 1$ , go to (4), otherwise go to (1).
- (4) Return the number  $-x \log(u_1 u_2)$  as the  $Gamma(a,1)$  random number.

In simulating a sample path of a gamma process  $G = \{G_t, t \geq 0\}$  where  $G_t \sim Gamma(at, b)$ , we simulate the value of this process at time points  $\{n\Delta t, n = 0, 1, \dots\}$  by:

- Generating independent  $Gamma(a\Delta t, b)$  random numbers  $\{\mathcal{G}_n, n \geq 1\}$  using Berman’s generator. (For small  $\Delta t$ ,  $a\Delta t$ , will be smaller than 1, hence we can use Berman’s generator),
- Then

$$G_0 = 0, G_{n\Delta t} = G_{(n-1)\Delta t} + \mathcal{G}_n, n \geq 1 \tag{A1}$$

**A.1.2 SIMULATING A STANDARD BROWNIAN MOTION**

The simulation of a standard Brownian motion,  $W = \{W_t, t \geq 0\}$  is simplified by the observation that the process has normally distributed independent increments.

- Generate a series of standard normal random numbers  $\{v_n, n = 1, 2, \dots\}$  using the Box-Muller or inverse transformation method and for very small  $\Delta t$ ,
- Then for time points  $\{n\Delta t, n = 0, 1, \dots\}$  we have:

$$W_0 = 0, W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{\Delta t} v_n, n \geq 1. \quad (\text{A2})$$

With the above two simulated processes, sample paths of a VG process can then be easily obtained.

## A.2 Monte Carlo Pricing under the Variance-Gamma Model

The following procedure as discussed in Schoutens (op. cit.) is followed (the accuracy of the final estimate is a function of the sample paths chosen):

- (1) Calibrate the model on the available market data based on some measure of fit.
- (2) Using (1),
  - (2.1) Simulate a significant number,  $m$ , of paths of the stock-price process  $S = \{S_t, 0 \leq t \leq T\}$  by simulating the log price process via a simulation of the time-changing process:
    - (2.1.1) Simulate the rate of time change process  $y = \{y_t, 0 \leq t \leq T\}$ ,
    - (2.1.2) Calculate from (2.1.1) the time change  $Y = \left\{ Y_t = \int_0^t y_s ds, 0 \leq t \leq T \right\}$ ,
    - (2.1.3) Simulate the VG process  $X = \{X_t, 0 \leq t \leq Y_T\}$  sampling over the period  $[0, Y_T]$ ,
    - (2.1.4) Calculate the time-changed VG process  $X_{Y_t}, t \in [0, T]$ ,
    - (2.1.5) Calculate the stock-price process  $S = \{S_t, 0 \leq t \leq T\}$ .
  - (2.2) For each path  $i$ , calculate the payoff function  $g_i = G(\{S_u, 0 \leq u \leq T\})$ .

- (3) Calculate the mean of the sample payoffs to get an estimate of the expected payoff:

$$\hat{g} = \frac{1}{m} \sum_{i=1}^m g_i. \quad (\text{A3})$$

- (4) Discount the estimated payoff at the risk-free rate to get an estimate of the value of the embedded option:

$$\text{Value of derivative} = e^{-rT} \hat{g} \quad (\text{A4})$$

The plot and snapshot (Figures A1 and A2) show the simulated stock price/index values under the Variance-Gamma framework of 10 000 simulations.

## A.3 Fitting the Variance-Gamma Model

A.3.1 The calibration of the model may be carried out using log-returns data (see Seneta, 2004, and Cepni et al., 2013) or implicitly using option prices where the option-pricing

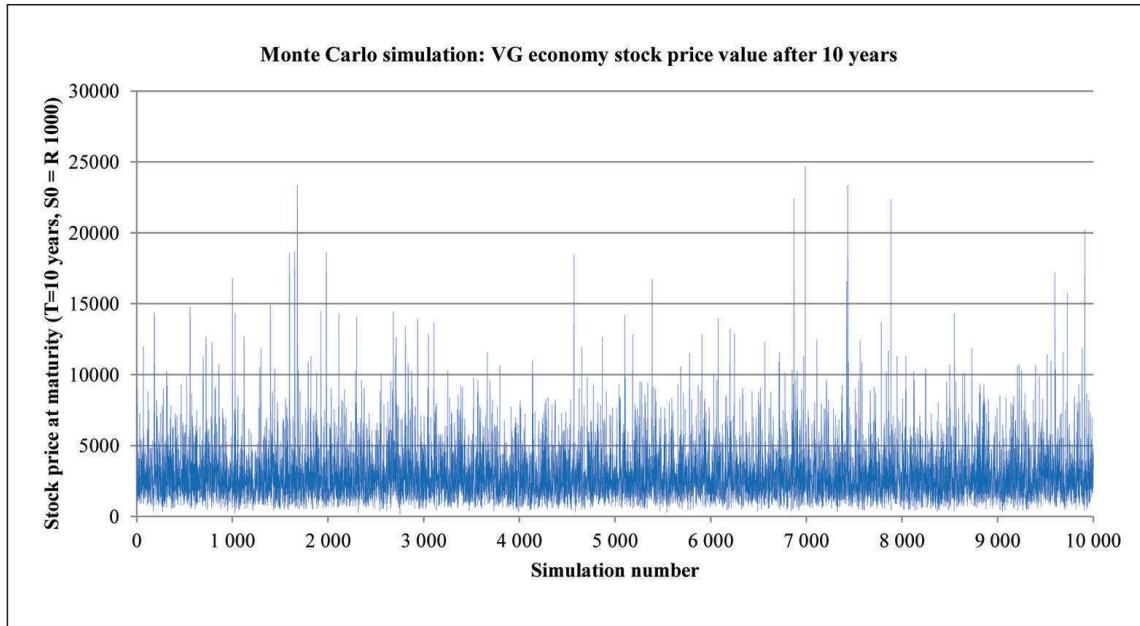


FIGURE A1. Simulated 10-year stock index values in a VG economy

Snapshot
2331.66638
2138.38232
932.230281
524.61483
3079.0495
1209.35685
2100.73155
2998.61262
5791.4007
2340.16209
1108.13029
2737.07275
1008.83435
6419.32653

FIGURE A2. Snapshot of simulated 10-year VG stock index values

formula is based on the characteristic function of the log price process of the stock. The latter is considered a more robust approach since it identifies parameters implied in the option prices which are more representative of future expectations.

A.3.2 However, the use of implied parameters should not come with a complete disregard of the valuable information that lies veiled in historical data. Implied parameters must be

compared with historical parameters and best estimates of the parameters arrived at based on these two pools of information.

A.3.3 In the absence of long-tenor options to estimate implied parameters in South Africa (SA), as would be needed for embedded options in a life insurance context, two possibilities exist. The first option is to calibrate the VG model based on United States of America (US) index options, such as S&P500 index options, which would be useful for the long tenors due to the presence of a deeper market. However, this approach is heavily dependent on the assumption that there is a link in behaviour between the US and SA markets. The US is a developed market whereas SA is an emerging market so there is no reason to believe that a steady relationship between the corresponding parameters exists. The second possibility is the use of historical data to calibrate the VG model.

A.3.4 Hardy (op. cit.) discusses the danger of ignoring historical data in the absence of any other information. The author notes that United Kingdom (UK) insurers in the 1980s and 1990s decided to make assumptions that ignored historical information; this led to the collapse of one company and serious difficulty for two others. If the historical information had been used in the parameterisation, the parameters so obtained would have captured the extremities that led to the difficulties in which the insurers found themselves and they could have avoided the corporate challenges that ensued.

A.3.5 In the context of variable annuity contracts, which in most cases rely heavily on the stock market returns, Hardy (op. cit.: 175) concludes that we cannot let “modelling fall into disrepute because we ‘cannot know’ whether the past is an adequate representation of the future.” Objective historical data should thus not be ignored when modelling long-term contracts on the assumption that they may not follow a process similar to the corresponding process of preceding years. It is from this basis that the VG model is calibrated for the case at hand.

A.3.6 The use of such an approach is affirmed if there is reason to believe that the historical observation is likely to be repeated in the pricing period to which the results are to be applied. This is the case in the research at hand, given the complete cycles observed in the historical period used. Ulmer (op. cit.) notes that this is comparable to making a choice on the risk-neutral distribution and does in some sense reflect the market’s choice of measure. Actuarial guidance on the matter in SA is given by the APN110 which notes that, in the absence of suitable instruments to fully calibrate the model, probable market values may be used, provided the results do not deviate from market-consistent values.

A.3.7 The hypothesis is that the VG framework can be used in the pricing of the options and the report will test this hypothesis for the SA context. A comparison is made with the RS model results contained in Ngugi, Maré & Kufakunesu (op. cit.).

A.3.8 Cepni et al. (op. cit.) fitted the VG model to a number of developed and emerging markets in assessing its adequacy in comparison with Normal distribution, the Normal Inverse Gaussian (NIG) and the Heston model. In their research, which includes South Africa, they conclude that the VG model performs better than all the other models they researched when daily and weekly returns are considered for the twenty countries in the study. The authors, however, do not discuss the estimation of parameters nor show the parameters derived from the log-returns for their models.

### A.3.9 Maximum-Likelihood Estimation of the VG Model

A.3.9.1 Estimation of the VG model is based on the log-returns data. The data used is once again based on the end-of-month ALSI closing values. The log-returns are derived and the plot for the period from July 1994 to June 2013 is shown in Figure A3 below.

A.3.9.2 The parameters may be estimated using the method of maximum-likelihood where, if we have a series of independent log-returns with  $\pi = (\sigma, \nu, \theta)$ , the VG probability density function parameters, then we find the set of parameters  $\pi$  that will maximise the logarithm of the likelihood function  $\mathcal{L}(\pi) = \prod_{i=1}^x f(x_i; \pi)$  where the  $x_i$  represents the series of the independent log-returns as in Brigo et al. (2007).

A.3.9.3 The authors discuss the central moments in a VG context and note that if we have sample estimates of the mean, variance, skewness and kurtosis of the log-returns denoted by  $M$ ,  $V$ ,  $S$  and  $K$  respectively, then the parameters can be initially estimated, using the method of moments, as:

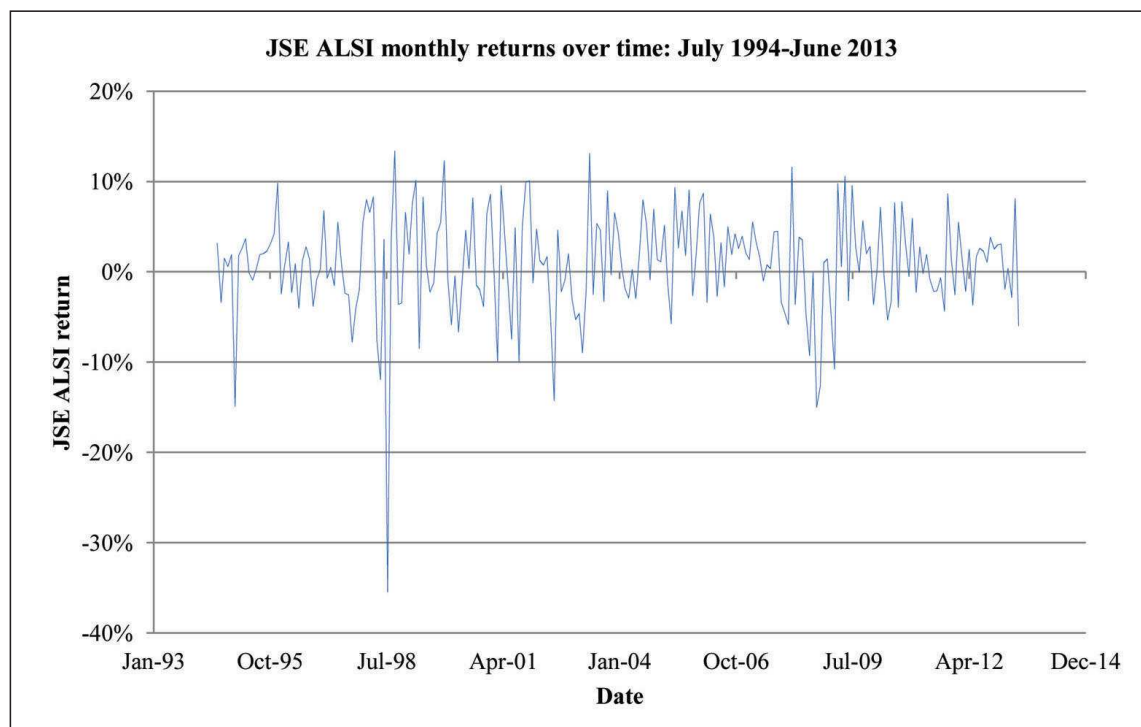


FIGURE A3. ALSI monthly returns time series: July 1994–June 2013

$$\begin{aligned} \bar{\sigma} &= \sqrt{\frac{V}{\Delta t}}, \\ \bar{\nu} &= \left(\frac{K}{3} - 1\right)\Delta t, \\ \bar{\theta} &= \frac{S\bar{\sigma}\sqrt{\Delta t}}{3\nu}, \\ \bar{\mu} &= \frac{\bar{X}}{\Delta t} - \bar{\theta}. \end{aligned} \tag{A5}$$

A.3.9.4 A comment regarding the crucial assumption of the independence of returns noted above is necessary. This assumption may be checked using the ARIMA procedure in the Statistical Analysis Software (SAS). It is satisfied if there is no autocorrelation in the historical returns. The ARIMA procedure shows no significant lags in the autocorrelations and partial autocorrelations as shown in Figures A4 and A5, so the independence assumption required in an MLE is satisfied.

A.3.9.5 This analysis also suggests the logarithmic return for the index to be a suitable choice for a market invariant and can thus be regarded as a source of uncertainty.

A.3.9.6 This estimation procedure is applied to the data with an easily scalable  $\Delta t$  of 1 from which we obtain the initial estimates as shown in Table A1.

A.3.9.7 These estimates are then used as the initial estimates in the MLE to fit the ALSI monthly log-returns. We obtain the final MLE VG parameter estimates based on the monthly returns as shown in Table 2.

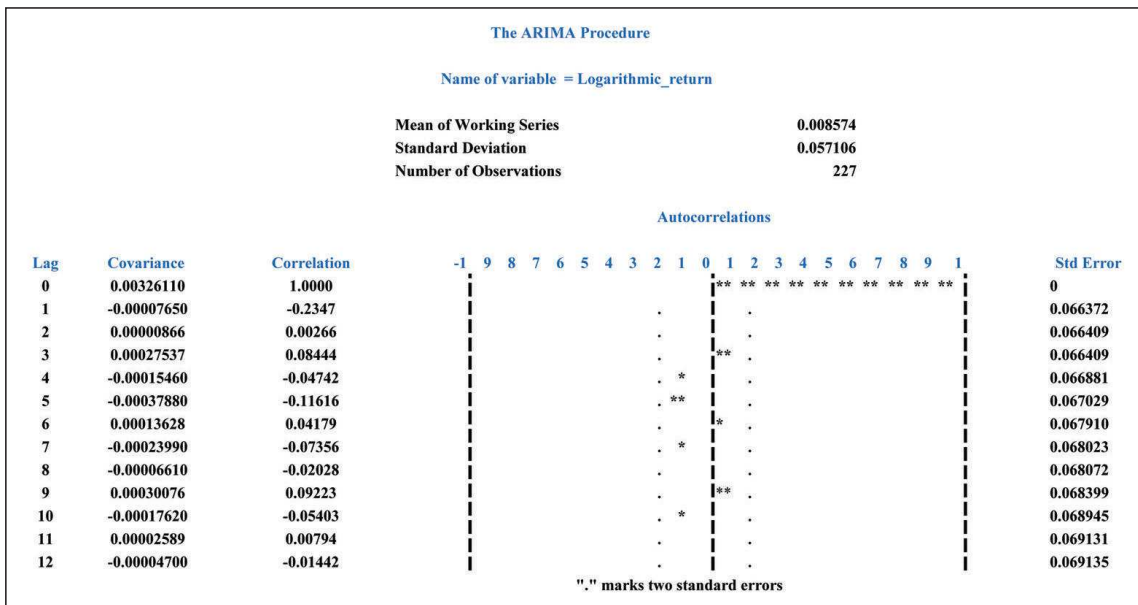


FIGURE A4. ALSI monthly returns autocorrelation plot

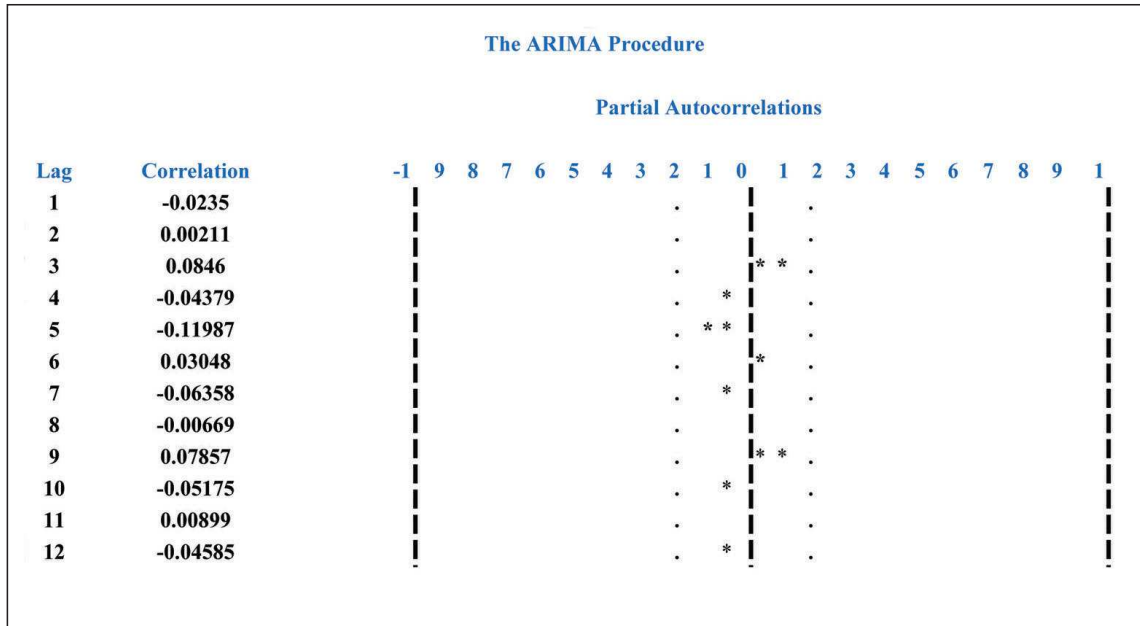


FIGURE A5. ALSI monthly returns partial autocorrelation plot

TABLE A1. Initial MME VG parameter estimates

Parameter	Mean (M)	Variance (V)	Skewness (S)	Kurtosis (K)
Estimate	0,008 573 8	0,003 275 5	-1,3049	9,544
Parameter	$\bar{\sigma}$	$\bar{\nu}$	$\bar{\theta}$	$\bar{\mu}$
Estimate	0,057 232	2,1813	-0,011 413	0,019 987

A.3.9.8 The parameters  $\theta$  and  $\nu$ , that are useful in controlling the skewness and kurtosis respectively, are non-zero. This is indicative of the need for the use of a framework other than the normal distribution. The VG presents itself as a credible alternative. The  $\theta$  parameter is negative which is indicative of negative skewness for the ALSI returns distribution.

A.3.9.9 The histogram plot (Figure A6) shows the extent of skewness and kurtosis against the super-imposed red curve being the normal distribution curve. In the South African context, the assumption of normality in returns for the index does not hold.

A.3.9.10 The QQ-plot in Figure A7 indicates the presence of tails that are thicker than in the normal distribution case, therefore the log-returns are heavily kurtotic.

A.3.9.11 The goodness-of-fit tests for normality shown in Figure A8 all lead to a rejection of the normality assumption for the returns at a 1% level of significance with the  $p$ -values for all three tests being less than 0,010.

A.3.9.12 This further adds credence to the use of a VG economy in the pricing of options for the South African context. Wang (op. cit.) notes that for such pricing (and hedging), the use of a probability distribution that correctly and accurately reflects the underlying asset’s behaviour is critical. The adoption of the VG framework in the particular

context at hand, namely embedded options, flows naturally from this, given the investment of most of the life insurers' funds in the stock markets.

A.3.9.13 The moments obtained are estimated from monthly log-return data and for ease of interpretation are changed to annualised form in Table A2 below. Backus, Foresi &

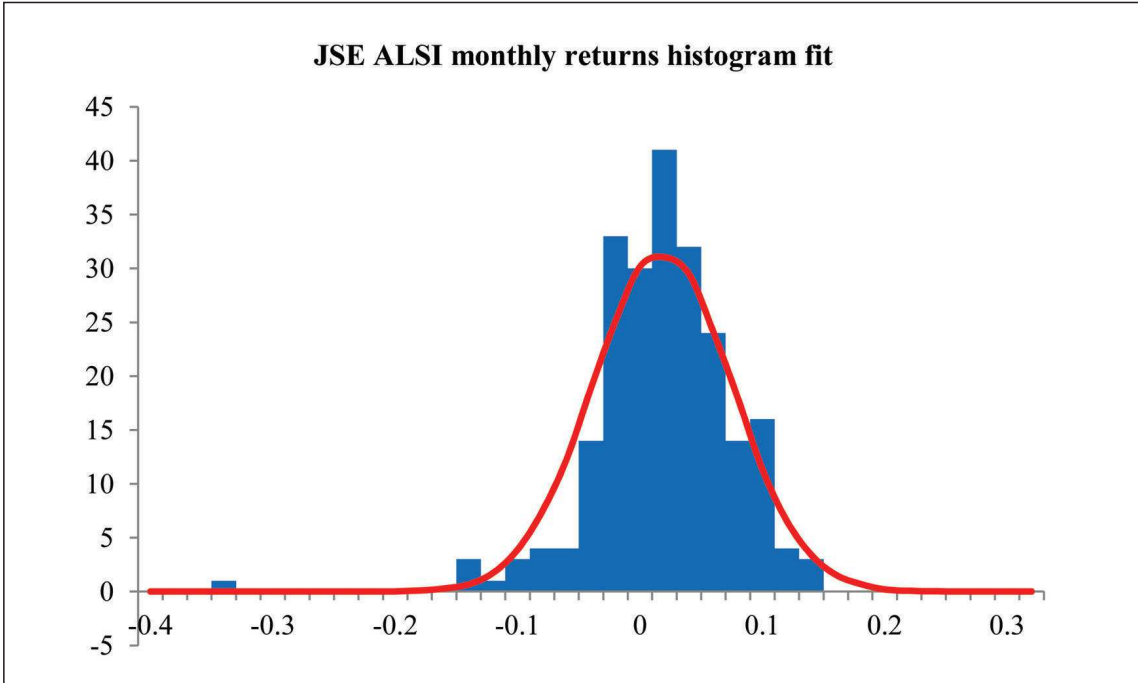


FIGURE A6. ALSI monthly returns histogram fit

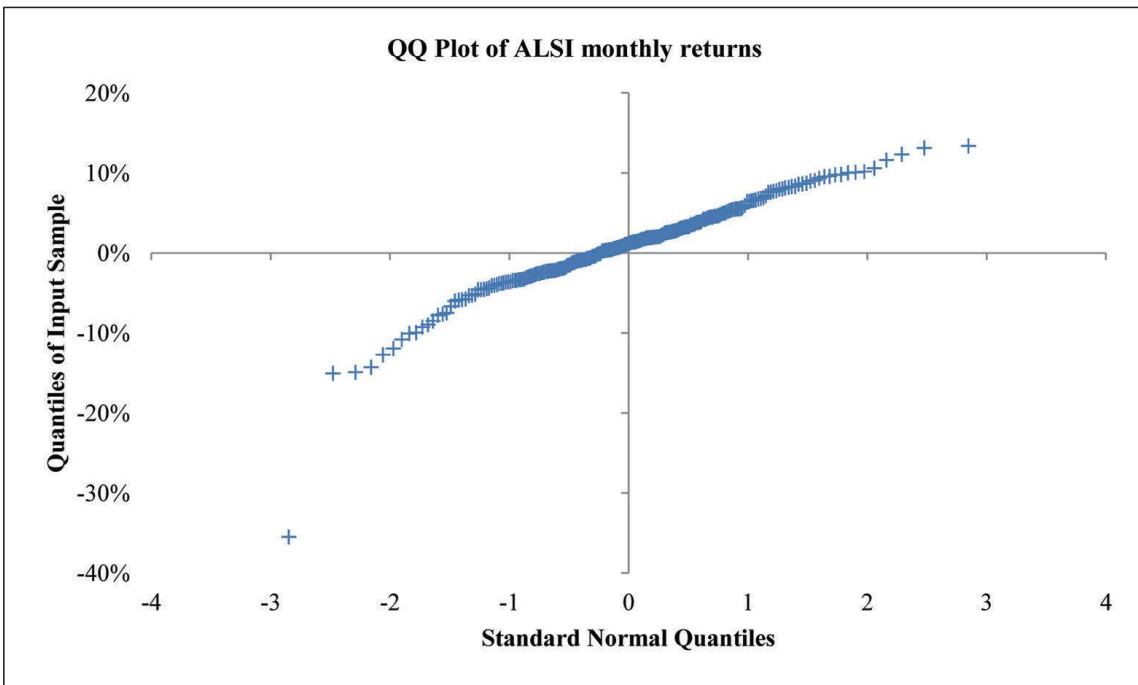


FIGURE A7. QQ plot of the ALSI monthly returns

Wu (2004) note that if the one-period skewness and excess kurtosis is denoted by  $S$  and  $K$  respectively, then the  $T$ -period skewness and excess kurtosis are given by  $S^* = \frac{S}{\sqrt{T}}$  and  $K^* = \frac{K}{T}$ , respectively.

TABLE A2. Estimated statistical moments on the ALSI returns

	Mean	Std dev	Skewness	Excess kurtosis
Monthly rate	0,008 574	0,057 232	-1,3049	6,544
Annualised rate	0,102 886	0,198 257	-0,3767	0,5453

The pricing calculations are, however, carried out using a month as the time unit.

**A.3.10 Arbitrage-Freeness, Parameter Risk-Neutrality, Stability and Robustness**

A.3.10.1 The VG model belongs to the so-called exponential Lévy models. Cont & Tankov (2004) show that the VG model is arbitrage-free since its trajectories, being an exponential Lévy model, are neither increasing nor decreasing with probability 1. It thus satisfies a key expectation in quantitative finance, that a pricing model should be arbitrage-free.

A.3.10.2 The pricing of options is completed in the risk-neutral world and it is critical that the parameters used in the pricing be the risk-neutral parameters. Madan, Carr & Chang (op. cit.) note that, unlike diffusion-based price processes, for a VG model, the statistical parameter estimates need not be equal to the risk-neutral parameters. The danger with using statistical parameters in this setting is that the risk premium that is incorporated in these risk-neutral parameters, partly due to market incompleteness, may not be incorporated in the statistical parameter estimation approach. A parameter sensitivity analysis can help in gauging the possible impact of this on the prices so obtained.

A.3.10.3 Madan, Carr & Chang (op. cit.) note that if one takes the risk-neutral parameters as constant across time then pooled time-series data can be used to estimate the statistical and risk-neutral processes. This approach is more necessary when sufficient data

Goodness-of-Fit Tests for Normal Distribution				
Test		----Statistic----		-----p Value-----
Kolmogorov-Smirnov	D	0.07112887	Pr > D	<0.010
Cramer-von Mises	W-Sq	0.21243629	Pr > W-Sq	<0.005
Anderson-Darling	A-Sq	1.50139543	Pr > A-Sq	<0.005

FIGURE A8. Goodness-of-fit tests for the ALSI monthly returns

are not available but the possibility of price discrepancies between the real option prices and those calculated using the parameters obtained still remains.

A.3.10.4 In the analyses performed in this paper, the use of Monte Carlo simulation instead of analytical pricing formulas avoids the possibility of excessive price estimation error due to the parameter estimation methodology. The VG process is calibrated using the MLE approach and future ALSI scenarios are built. The options under consideration are vanilla options so the payoff is calculated at maturity and discounted using the continuously compounded risk-free interest rate. The risk-free interest rate is readily available and does not introduce major discrepancies. The credibility of the approach is discussed in the paper by Embrechts (op. cit.) who notes that, in the context of insurance pricing, the method gives a more objective description of the underlying randomness.

A.3.10.5 In the process of calibration, great care has to be taken to ensure that the model parameters so obtained not only match the observed market prices closely but that they are also stable.

A.3.10.6 The 95% confidence intervals for the parameters are contained in Table A3 together with the standard errors. The standard errors are all reasonably low with the highest standard error being on the estimated  $\nu$  parameter; thus 95% of the time, if we assume a VG economy for the ALSI and fit using the log-returns, the parameters obtained will fall within the confidence intervals shown.

TABLE A3. Statistical significance of the VG estimated parameters

Parameter	$\theta$	$\nu$	$\sigma$	$\mu$
LCI	-0,0406	0,0210	0,0474	-0,0015
UCI	0,0111	0,8711	0,0613	0,0482
Standard error	0,013 197	0,216 873	0,003 565	0,012 677

### A.3.11 VG Parameter Sensitivity Analysis

A.3.11.1 The lack of long-tenor options from which the VG model should be calibrated in South Africa means that the parameters must be tested to assess the impact on the price if they change from the base case. This crucial sensitivity analysis on the 10-year European put option (with both  $S_0$  and strike price R1000) to changes in the three VG parameters is shown in Table 4.

A.3.11.2 The European put options at long maturities under the VG economy appear to be quite sensitive to changes in the volatility parameter, modestly sensitive to the skewness parameter and least sensitive to the kurtosis parameter.

A.3.11.3 As volatility increases, the price increases, significantly so. This intuitively makes sense since in a more volatile economy (higher volatility) the prices are expected to be higher as noted in Ngugi, Maré & Kufakunesu (op. cit.). The put option price appears to be an increasing function of the kurtosis parameter.

A.3.11.4 As the theta parameter becomes more negative, which is indicative of greater negative skewness, the prices increase, signalling a skewness premium. From the

base scenario to a 1% increase, the price decreases. When the tweak is 5% the price increases over the base scenario. This makes intuitive sense since theta is a measure of skewness. As we move from the base scenario and add a positive tweak, the skewness decreases as reflected in a price decrease. A theta parameter of 0 (a tweak of about +1,5%) yields a price of 5,2124 in which case we are working with the symmetric VG case discussed in Madan & Seneta (op. cit.) which yields the lowest price. After this, positive skewness sets in and the price starts to increase to adjust the skewness premium.

A.3.11.5 It is also worthwhile to note that the analysis above, with an adjustment of  $\pm 5\%$  on the parameters, will lie outside the 95% confidence intervals in Table A3 other than for the  $\nu$  parameter. However, the prices do not show significant sensitivity to the  $\nu$  parameter, hence the overall statistical confidence from the above results is above the commonly accepted 95%. The only concern would then be the correlation risk since, in practice the parameters do not change independently and a change in one parameter could have a ripple effect on the rest. The issue of correlation is, however, outside the scope of this research.

#### A.4 MODIFIED BESSEL FUNCTION OF THE SECOND KIND

The notation normally used for this function is  $K$  but to avoid confusion with the usage of  $K$  to denote the strike price, the symbol  $\Psi$  is used in this paper. The function is closely related to the modified Bessel function of the first kind discussed in Ngugi, Maré & Kufakunesu (op. cit.) and is expressed here below in terms of this function.

$$\Psi_n(x) = \frac{\pi I_{-n}(x) - I_n(x)}{2 \sin(n\pi)}$$

#### A.5 DEGENERATE HYPERGEOMETRIC FUNCTION

The degenerate hypergeometric function has a hypergeometric series given by

$${}_1F_1(a; b; z) = 1 + \frac{a}{b}z + \frac{a(a+1)}{b(b+1)} \frac{z^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$$

where  $(a)_k$  and  $(b)_k$  are the Pochhammer symbols.

#### A.6 ASSA MORTALITY TABLES

The mortality rates used in this paper, Tables A4 and A5 below, are derived from the ASSA 2008 national population model for the year 2013, the SA85-90 Assured Lives Mortality table and the SAIFL 98 and SAIML 98 standard tables discussed in Dorrington & Tootla (2007).

TABLE A4. SAIML98 and SAIFL98 standard table of mortality rates

Age (x)	Males	Females
	${}_1q_x$	${}_1q_x$
50	0.00628	0.00283
55	0.00979	0.00442
60	0.01536	0.00694
65	0.0234	0.01105
70	0.03319	0.0174

TABLE A5. ASSA 2008 10-year survival and mortality rates

Age	Males		Females	
	${}_{10}P_x$	${}_{10}q_x = 1 - {}_{10}P_x$	${}_{10}P_x$	${}_{10}q_x = 1 - {}_{10}P_x$
45	0.67704	0.32296	0.74243	0.25757
50	0.58828	0.41172	0.63710	0.36290
55	0.50778	0.49222	0.54613	0.45387
60	0.45722	0.54278	0.54441	0.45559
65	0.40762	0.59238	0.54970	0.45030
70	0.30741	0.69259	0.43839	0.56161
75	0.26982	0.73018	0.43108	0.56892